Character Theory

Lecture notes winter semester 2013/14 and summer semester 2018

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	p	power	A	A	A	A	A	A	B	A	AA	BB	A	A	B	AB	A	AA	BC	AA	BA	AA	AA	BB	AB	A	A	
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	ind	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	B**	8A	10A	11A	12A	12B	14A	B**	15A	B**	21A	B**	23A	B**	
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	χ_1
χ_2	+	23	7	−1	5	−1	−1	3	−1	3	1	−1	2	2	1	−1	1	−1	−1	0	0	0	0	−1	−1	0	0	χ_2
χ_3	o	45	−3	5	0	3	−3	1	1	0	0	−1	b7	**	−1	0	1	0	1	−b7	**	0	0	b7	**	−1	−1	χ_3
χ_4	o	45	−3	5	0	3	−3	1	1	0	0	−1	**	b7	−1	0	1	0	1	**	−b7	0	0	**	b7	−1	−1	χ_4
χ_5	o	231	7	−9	−3	0	−1	−1	3	1	1	0	0	0	−1	1	0	−1	0	0	0	b15	**	0	0	1	1	χ_5
χ_6	o	231	7	−9	−3	0	−1	−1	3	1	1	0	0	0	−1	1	0	−1	0	0	0	**	b15	0	0	1	1	χ_6
χ_7	+	252	28	12	9	0	4	4	0	2	1	0	0	0	0	2	−1	1	0	0	0	−1	−1	0	0	−1	−1	χ_7
χ_8	+	253	13	−11	10	1	−3	1	1	3	−2	1	1	1	−1	−1	0	0	1	−1	−1	0	0	1	1	0	0	χ_8
χ_9	+	483	35	3	6	0	3	3	3	−2	2	0	0	0	−1	−2	−1	0	0	0	0	1	1	0	0	0	0	χ_9
χ_{10}	o	770	−14	10	5	−7	2	−2	−2	0	1	1	0	0	0	0	0	−1	1	0	0	0	0	0	0	b23	**	χ_{10}
χ_{11}	o	770	−14	10	5	−7	2	−2	−2	0	1	1	0	0	0	0	0	−1	1	0	0	0	0	0	0	**	b23	χ_{11}
χ_{12}	o	990	−18	−10	0	3	6	2	−2	0	0	−1	b7	**	0	0	0	0	1	b7	**	0	0	b7	**	1	1	χ_{12}
χ_{13}	o	990	−18	−10	0	3	6	2	−2	0	0	−1	**	b7	0	0	0	0	1	**	b7	0	0	**	b7	1	1	χ_{13}
χ_{14}	+	1035	27	35	0	6	3	−1	3	0	0	2	−1	−1	1	0	1	0	0	−1	−1	0	0	−1	−1	0	0	χ_{14}
χ_{15}	o	1035	−21	−5	0	−3	3	3	−1	0	0	1	2b7	**	−1	0	1	0	−1	0	0	0	0	−b7	**	0	0	χ_{15}
χ_{16}	o	1035	−21	−5	0	−3	3	3	−1	0	0	1	**	2b7	−1	0	1	0	−1	0	0	0	0	**	−b7	0	0	χ_{16}
χ_{17}	+	1265	49	−15	5	8	−7	1	−3	0	1	0	−2	−2	1	0	0	−1	0	0	0	0	0	1	1	0	0	χ_{17}
χ_{18}	+	1771	−21	11	16	7	3	−5	−1	1	0	−1	0	0	−1	1	0	0	−1	0	0	1	1	0	0	0	0	χ_{18}
χ_{19}	+	2024	8	24	−1	8	8	0	0	−1	−1	0	1	1	0	−1	0	−1	0	1	1	−1	−1	1	1	0	0	χ_{19}
χ_{20}	+	2277	21	−19	0	6	−3	1	−3	−3	0	2	2	2	−1	1	0	0	0	0	0	0	0	−1	−1	0	0	χ_{20}
χ_{21}	+	3312	48	16	0	−6	0	0	0	−3	0	−2	1	1	0	1	1	0	0	−1	−1	0	0	1	1	0	0	χ_{21}
χ_{22}	+	3520	64	0	10	−8	0	0	0	0	−2	0	−1	−1	0	0	0	0	0	1	1	0	0	−1	−1	1	1	χ_{22}
χ_{23}	+	5313	49	9	−15	0	1	−3	−3	3	1	0	0	0	−1	−1	0	1	0	0	0	0	0	0	0	0	0	χ_{23}
χ_{24}	+	5544	−56	24	9	0	−8	0	0	−1	1	0	0	0	0	−1	0	1	0	0	0	−1	−1	0	0	1	1	χ_{24}
χ_{25}	+	5796	−28	36	−9	0	−4	4	0	1	−1	0	0	0	0	1	−1	−1	0	0	0	1	1	0	0	0	0	χ_{25}
χ_{26}	+	10395	−21	−45	0	0	3	−1	3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	−1	−1	χ_{26}

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Warning: This is an AI-translated version of my German lectures notes, performed by *Gemini 3 Flash Preview*. I have not checked whether Gemini introduced errors. Use with care!

Preface

The present notes are based on lectures in the winter semester 2013/14 and in the summer semester 2018 at the Friedrich Schiller University in Jena. These were $3 + 1$ lectures for the Master's program in mathematics. My goal was to prove the main theorems of the character theory of finite groups as quickly and elementarily as possible. Only prior knowledge of Algebra 1 was required from the audience (elementary group theory and some Galois theory). I have consistently avoided the terms "module" and "group algebra" (usually components of an Algebra 2 lecture). I would like to thank René Reichenbach and Sebastian Uschmann for numerous error reports and suggestions for improvement.

The following sources form the basis of the notes (in descending priority):

- Külshammer, Skript zur Darstellungstheorie
- Isaacs, *Character theory of finite groups*, AMS Chelsea Publishing, Providence, RI, 2006
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1 Representations and Characters

Let G always be a finite group.

Definition 1.1. Let $V \neq 0$ be a finite-dimensional complex vector space. A *representation* of G is a homomorphism $\Delta\colon G \to \mathrm{GL}(V)$. The *degree* of the representation is $n := \dim V$. By choosing a basis of V, one obtains a corresponding *matrix representation* $\Delta' : G \to \mathrm{GL}(n, \mathbb{C})$.

Example 1.2.

(i) The *trivial* (matrix) representation $1_G : G \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^\times$ is given by $g \mapsto 1$ for $g \in G$.

(ii) For $n \in \mathbb{N}$, the map $\mathrm{sgn} : S_n \to \mathbb{C}^\times$, $g \mapsto \mathrm{sgn}(g)$ is a representation of degree 1.

(iii) For two representations $\Delta\colon G \to \mathrm{GL}(V)$ and $\Gamma\colon G \to \mathrm{GL}(W)$, the map $\Delta \oplus \Gamma\colon G \to \mathrm{GL}(V \times W)$ is also a representation. Here, $((\Delta \oplus \Gamma)(g))(v, w) := ((\Delta(g))(v), (\Gamma(g))(w))$ for $g \in G$, $v \in V$ and $w \in W$.

(iv) If $\Delta\colon G \to \mathrm{GL}(V)$ is a representation and $H \leq G$, then one obtains by restriction a representation $\Delta_H : H \to \mathrm{GL}(V)$, $h \mapsto \Delta(h)$.

(v) If $N \trianglelefteq G$ and $\Delta\colon G/N \to \mathrm{GL}(V)$ is a representation, then one obtains by *inflation* a representation $G \to \mathrm{GL}(V)$, $g \mapsto \Delta(gN)$ on G. We will often denote this also by Δ.

(vi) If $\Delta\colon G \to \mathrm{GL}(V)$ is a representation and $N \trianglelefteq G$ with $N \subseteq \mathrm{Ker}(\Delta)$, then one obtains by *deflation* a well-defined representation $\widehat{\Delta} : G/N \to \mathrm{GL}(V)$, $gN \mapsto \Delta(g)$. In particular, $\widehat{\Delta} : G/\mathrm{Ker}(\Delta) \to \mathrm{GL}(V)$ is a *faithful* representation, i. e. $\widehat{\Delta}$ is injective.

(vii) Inflation and deflation are obviously inverse to each other.

Definition 1.3. Two representations $\Delta\colon G \to \mathrm{GL}(V)$ and $\Gamma\colon G \to \mathrm{GL}(W)$ are called *similar*, if an isomorphism $f : V \to W$ with $f \circ \Delta(g) = \Gamma(g) \circ f$ exists for all $g \in G$. If applicable, the following diagram is thus commutative:

$$\begin{array}{ccc} V & \xrightarrow{\Delta(g)} & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{\Gamma(g)} & W \end{array}$$

Accordingly, two matrix representations $\Delta\colon G \to \mathrm{GL}(n, \mathbb{C})$ and $\Gamma\colon G \to \mathrm{GL}(m, \mathbb{C})$ are similar if $n = m$ and there exists an $A \in \mathrm{GL}(n, \mathbb{C})$ with $A\Delta(g) = \Gamma(g)A$ for all $g \in G$.

Remark 1.4.

(i) Similar representations have the same degree.

(ii) Similarity is an equivalence relation.

(iii) One is usually only interested in representations up to similarity (just as for groups up to isomorphism).

(iv) In linear algebra, it is shown that two square matrices A, B describe the same map if and only if there exists an invertible matrix T with $AT = TB$. Thus, two matrix representations Γ_1 and Γ_2 that correspond to a fixed representation Δ of G are always similar.

(v) The similarity classes of representations and matrix representations obviously correspond to each other. We will therefore often identify representations with their corresponding matrix representations in the following.

Definition 1.5. Let $\Delta\colon G \to \mathrm{GL}(V)$ be a representation. A subspace $U \leq V$ is called Δ-*invariant*, if $(\Delta(g))(u) \in U$ holds for all $g \in G$ and $u \in U$. If applicable, $\Delta' : G \to \mathrm{GL}(U)$, $g \mapsto \Delta(g)_{|U}$ is also a representation. If 0 and V are the only Δ-invariant subspaces, then Δ is *irreducible*. Otherwise, Δ is *reducible*.

Example 1.6.

(i) Representations of degree 1 are obviously irreducible.

(ii) Inflation and deflation of irreducible representations are again irreducible (the images do not change).

Theorem 1.7 (Maschke)**.** *Let $\Delta\colon G \to \mathrm{GL}(V)$ be a representation and $U \leq V$ be Δ-invariant. Then U possesses a Δ-invariant complement $W \leq V$, i. e. $V = U \oplus W$.*

Proof. We first choose an arbitrary subspace X of V with $V = U \oplus X$ (linear algebra) and denote by $h : V \to V$ the corresponding projection onto U. Then we set

$$g := \frac{1}{|G|} \sum_{x \in G} \Delta(x^{-1}) \circ h \circ \Delta(x)$$

and $W := \mathrm{Ker}(g)$. For $u \in U$ we thus have

$$g(u) = \frac{1}{|G|} \sum_{x \in G} (\Delta(x^{-1}) \circ h \circ \Delta(x))(u) = \frac{1}{|G|} \sum_{x \in G} \underbrace{(\Delta(x^{-1}) \circ \Delta(x))}_{=\Delta(x^{-1}x)=\Delta(1)=\mathrm{id}_V} (u) = u.$$

In particular, $U \cap W = 0$. For $v \in V$ we have $g(v) \in U$, so

$$g(v - g(v)) = g(v) - g(g(v)) = g(v) - g(v) = 0,$$

i. e. $v - g(v) \in W$ and $v = g(v) + (v - g(v)) \in U + W$. Consequently, $V = U \oplus W$. For $w \in W$ and $y \in G$ we have

$$\begin{aligned}(g \circ \Delta(y))(w) &= \left(\frac{1}{|G|} \sum_{x \in G} \Delta(x^{-1}) \circ h \circ \Delta(xy)\right)(w) \\ &= \left(\Delta(y) \circ \underbrace{\left(\frac{1}{|G|} \sum_{x \in G} \Delta(y^{-1}x^{-1}) \circ h \circ \Delta(xy)\right)}_{=g}\right)(w) \\ &= (\Delta(y) \circ g)(w) = (\Delta(y))(0) = 0,\end{aligned}$$

thus $(\Delta(y))(w) \in \mathrm{Ker}(g) = W$. Consequently, W is Δ-invariant. □

Remark 1.8. Let Δ be a representation on V, and let $V = U \oplus W$ be a Δ-invariant decomposition. This yields subrepresentations $\Gamma_U : G \to \mathrm{GL}(U)$, $g \mapsto \Delta(g)_{|U}$ and $\Gamma_W : G \to \mathrm{GL}(W)$, $g \mapsto \Delta(g)_{|W}$. By choosing a suitable basis of V, Δ then has the form

$$\Delta(g) = \begin{pmatrix} \Gamma_U(g) & 0 \\ 0 & \Gamma_W(g) \end{pmatrix}$$

for all $g \in G$. Thus $\Delta = \Gamma_U \oplus \Gamma_W$. Every representation can therefore be written as a direct sum of irreducible representations.

Lemma 1.9 (SCHUR's Lemma)**.** *Let $\Delta\colon G \to \mathrm{GL}(n,\mathbb{C})$, $\Gamma\colon G \to \mathrm{GL}(m,\mathbb{C})$ be irreducible matrix representations and $0 \neq A \in \mathbb{C}^{n\times m}$ with $A\Gamma(g) = \Delta(g)A$ for all $g \in G$. Then $n = m$ and A is invertible (in particular, Δ and Γ are similar). In the case $\Delta = \Gamma$, $A = \lambda 1_n$ holds for some $\lambda \in \mathbb{C}^\times$.*

Proof. For $g \in G$ and $v \in \mathrm{Ker}(A)$, we have

$$(A\Delta(g))v = (\Gamma(g)A)v = 0,$$

so $(\Delta(g))v \in \mathrm{Ker}(A)$. Therefore $\mathrm{Ker}(A)$ is a Δ-invariant subspace of $\mathbb{C}^m$. Analogously, $\mathrm{Im}(A)$ is a Γ-invariant subspace of $\mathbb{C}^n$. Thus $\mathrm{Ker}(A) = 0$ and $\mathrm{Im}(A) = \mathbb{C}^n$ due to the irreducibility of Δ and Γ. Consequently, A is invertible and $n = m$. Now let $\Delta = \Gamma$. Let λ be an eigenvalue of A. Then $(A - \lambda 1_n)\Gamma(g) = \Delta(g)(A - \lambda 1_n)$ also holds for all $g \in G$. Since $A - \lambda 1_n$ is not invertible, $A - \lambda 1_n = 0$ follows from the first part of the proof. □

Theorem 1.10. *Every irreducible representation of an abelian group has degree* 1*.*

Proof. Let G be abelian and $\Delta\colon G \to \mathrm{GL}(n,\mathbb{C})$ an irreducible matrix representation of G. Let $g \in G$ be fixed. For all $h \in G$, it then holds that $\Delta(g)\Delta(h) = \Delta(gh) = \Delta(hg) = \Delta(h)\Delta(g)$. According to Schur's Lemma, $\Delta(g) = \lambda_g 1_n$ for some $\lambda_g \in \mathbb{C}$. In particular, $\mathbb{C}(1, 0, \ldots, 0)$ is a Δ-invariant subspace of $\mathbb{C}^n$. Since Δ is irreducible, $n = 1$ follows. □

Definition 1.11. Let $\Delta\colon G \to \mathrm{GL}(n,\mathbb{C})$ be a matrix representation. The map $\chi\colon G \to \mathbb{C}$, $g \mapsto \mathrm{Trace}\,\Delta(g)$ is called the *character* of Δ (and of G). Here, $\chi(1) = \mathrm{Trace}\,\Delta(1) = \mathrm{Trace}\,1_n = n$ is the *degree* of χ (and of Δ). If Δ is irreducible (faithful, ...), then χ is also called *irreducible* (*faithful*, ...). We denote the set of irreducible characters of G by $\mathrm{Irr}(G)$.

Lemma 1.12. *Similar matrix representations have the same character.*

Proof. Let $\Delta\colon G \to \mathrm{GL}(n,\mathbb{C})$ and $\Gamma\colon G \to \mathrm{GL}(n,\mathbb{C})$ be similar matrix representations. Then there exists an $A \in \mathrm{GL}(n,\mathbb{C})$ with $\Delta(g)A = A\Gamma(g)$ for all $g \in G$. From linear algebra it is known that for square matrices M_1, M_2 of the same dimension: $\mathrm{Trace}(M_1M_2) = \mathrm{Trace}(M_2M_1)$. Thus $\mathrm{Trace}\,\Delta(g) = \mathrm{Trace}((A\Gamma(g))A^{-1}) = \mathrm{Trace}(A^{-1}(A\Gamma(g))) = \mathrm{Trace}\,\Gamma(g)$ for all $g \in G$. □

Remark 1.13.

(i) If $\Delta\colon G \to \mathrm{GL}(V)$ is a representation, one can assign a character to Δ by choosing a corresponding matrix representation. Because of Lemma 1.12, this does not depend on the choice of the basis of V.

(ii) Characters are the "shadows" of representations, i.e., on the one hand, information is lost by replacing the n^2 entries of a matrix with a single value, but on the other hand, enough information remains to read off properties of the group.

(iii) Obviously, representations of degree 1 coincide with their character. These characters are called *linear*. In particular, there is the *trivial* character $1_G : G \to \mathbb{C}$ with $1_G(g) = 1$ for $g \in G$.

(iv) If Δ and Γ are representations with character χ_Δ and χ_Γ respectively, then $\Delta\oplus\Gamma$ has the character $\chi_\Delta + \chi_\Gamma$. Sums of characters are thus characters again.

(v) For every representation $\Delta\colon G \to \mathrm{GL}(V)$, $\det\Delta\colon G \to \mathbb{C}$, $g \mapsto \det\Delta(g)$ is a character of degree 1.

Definition 1.14.

(i) Let $g \in G$. Then $C := \{hgh^{-1} : h \in G\}$ is called the *conjugacy class* of g. Obviously, $\{1\}$ is a conjugacy class of G. The set of conjugacy classes of G is $\mathrm{Cl}(G)$. If $h \in C$, then g and h are *conjugate*. Let $\mathrm{C}_G(g) := \{x \in G : xg = gx\} \le G$ be the *centralizer* of g in G. In Algebra 1 it is shown that $|C| = |G : \mathrm{C}_G(g)|$.

(ii) A map $f : G \to \mathbb{C}$ is called a *class function* if $f(g) = f(hgh^{-1})$ holds for all $g, h \in G$. Class functions are therefore constant on conjugacy classes.

Lemma 1.15.

(i) The characters of a group G are class functions.

(ii) The set $\mathrm{CF}(G)$ of class functions forms a $\mathbb{C}$-vector space via $(\alpha + \beta)(g) := \alpha(g) + \beta(g)$ and $(a \cdot \alpha)(g) = a\alpha(g)$ for $\alpha, \beta \in \mathrm{CF}(G)$, $a \in \mathbb{C}$ and $g \in G$. Here, $\dim \mathrm{CF}(G) = |\mathrm{Cl}(G)|$.

Proof.

(i) Let $\Delta\colon G \to \mathrm{GL}(V)$ be a representation with character χ. For $g, h \in G$ we have

$$\chi(hgh^{-1}) = \operatorname{Trace} \Delta(hgh^{-1}) = \operatorname{Trace}(\Delta(h)\Delta(g)\Delta(h)^{-1}) = \operatorname{Trace} \Delta(g) = \chi(g).$$

(ii) Trivial. □

Definition 1.16. Obviously,

$$(\chi, \psi)_G := \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)} \qquad (\chi, \psi \in \mathrm{CF}(G))$$

defines a scalar product on the $\mathbb{C}$-vector space $\mathrm{CF}(G)$. In this way, $\mathrm{CF}(G)$ becomes a Hilbert space.

Remark 1.17. For characters χ, ψ of G, according to Exercise 5, we also have

$$(\chi, \psi)_G = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$

Lemma 1.18 (Schur relations)**.** *Let $\Delta\colon G \to \mathrm{GL}(n, \mathbb{C})$, $\Gamma\colon G \to \mathrm{GL}(m, \mathbb{C})$ be irreducible matrix representations with $\Delta(g) = (\lambda_{ij}(g))$ and $\Gamma(g) = (\theta_{ij}(g))$ for $g \in G$.*

(i) If Δ and Γ are not similar, then

$$\sum_{g \in G} \lambda_{ii}(g)\theta_{jj}(g^{-1}) = 0$$

for all i, j.

(ii) We have

$$\sum_{g \in G} \lambda_{ii}(g)\lambda_{jj}(g^{-1}) = \frac{|G|}{n}\delta_{ij}.$$

Proof. Let $E_{ij} \in \mathbb{C}^{n\times m}$ be the matrix with a 1 at position (i,j) and zeros elsewhere. We set

$$F_{ij} := \sum_{g\in G} \Delta(g)E_{ij}\Gamma(g^{-1}).$$

For $h \in G$ we then have $\Delta(h)F_{ij}\Gamma(h^{-1}) = F_{ij}$, i.e., $\Delta(h)F_{ij} = F_{ij}\Gamma(h)$. If Δ and Γ are not similar, then $F_{ij} = 0$ follows from Schur's Lemma. In particular, F_{ij} is equal to 0 at position (i,j), i.e., (i) holds.

Now let $\Delta = \Gamma$. According to Schur, $F_{ij} = \rho_{ij} \cdot 1_n$ for some $\rho_{ij} \in \mathbb{C}$. For the entry of F_{ij} at position $(1,1)$, we then have

$$\rho_{ij} = \sum_{g\in G} \lambda_{1i}(g)\lambda_{j1}(g^{-1}) = \sum_{h\in G} \lambda_{1i}(h^{-1})\lambda_{j1}(h) = \sum_{h\in G} \lambda_{j1}(h)\lambda_{1i}(h^{-1}) = \rho_{11}\delta_{ij}.$$

With $\rho := \rho_{11}$ $(= \rho_{ii})$, we then have

$$n\rho = \sum_{j=1}^{n}\sum_{g\in G} \lambda_{ij}(g)\lambda_{ji}(g^{-1}) = \sum_{g\in G} 1 = |G|$$

due to $\Delta(g)\Delta(g^{-1}) = 1_n$ for $g \in G$. Now (ii) follows from the entry of F_{ij} at position (i,j). □

Theorem 1.19 (First Orthogonality Relation)**.** *For $\chi, \psi \in \mathrm{Irr}(G)$ we have*

$$\boxed{(\chi,\psi)_G = \frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\psi(g)} = \begin{cases}1 & \text{if } \chi = \psi,\\ 0 & \text{otherwise.}\end{cases}}$$

Proof. Let Δ and Γ be irreducible representations of G with character χ and ψ respectively. First let $\chi \neq \psi$. According to Lemma 1.12, Δ and Γ are then not similar. We write $\Delta(g) = (\lambda_{ij}(g))$ and $\Gamma(g) = (\theta_{ij}(g))$ for $g \in G$. Then $\chi(g) = \sum \lambda_{ii}(g)$ and $\psi(g) = \sum \theta_{ii}(g)$. According to Lemma 1.18 we thus have

$$(\chi,\psi)_G = \frac{1}{|G|}\sum_{g\in G}\sum_{i,j}\lambda_{ii}(g)\theta_{jj}(g^{-1}) = 0.$$

Analogously,

$$(\chi,\chi)_G = \frac{1}{|G|}\sum_{g\in G}\sum_{i,j}\lambda_{ii}(g)\lambda_{jj}(g^{-1}) = \frac{\chi(1)}{|G|}\frac{|G|}{\chi(1)} = 1.$$ □

Remark 1.20. From Theorem 1.19 it follows easily that $\mathrm{Irr}(G)$ is a linearly independent subset of $\mathrm{CF}(G)$. In particular, $|\mathrm{Irr}(G)| \le \dim_{\mathbb{C}} \mathrm{CF}(G) = |\mathrm{Cl}(G)| \le |G| < \infty$.

Theorem 1.21. *Two representations are similar if and only if they have the same character.*

Proof. One direction is Lemma 1.12. Now let Δ and Γ be representations with the same character χ. We write $\Delta = \bigoplus_{i=1}^{n} \Delta_i$ and $\Gamma = \bigoplus_{i=1}^{m} \Gamma_i$ as sums of irreducible representations. Then χ also decomposes into

$$\chi = \sum_{i=1}^{n} \chi_{\Delta_i} = \sum_{i=1}^{m} \chi_{\Gamma_i}.$$

According to Remark 1.20, $n = m$ and $\chi_{\Delta_i} = \chi_{\Gamma_i}$ given a suitable numbering. From Lemma 1.18 it now follows easily that Δ_i and Γ_i are similar. So let $A_i \in \mathrm{GL}(\chi_{\Delta_i}(1), \mathbb{C})$ with $A_i\Delta_i(g) = \Gamma_i(g)A_i$ for all $g \in G$. For

$$A := \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{pmatrix} \in \mathrm{GL}(\chi(1), \mathbb{C})$$

it then obviously holds that $A\Delta(g) = \Gamma(g)A$ for all $g \in G$, i.e. Δ and Γ are similar. □

Remark 1.22.

(i) Let ρ be the regular character of G. According to Exercise 2, it holds that

$$\rho(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $\chi \in \mathrm{Irr}(G)$, we therefore have

$$(\rho, \chi)_G = \frac{1}{|G|} \sum_{g \in G} \rho(g)\overline{\chi(g)} = \chi(1).$$

It follows that $\rho = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi$ and $\boxed{|G| = \rho(1) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2.}$

(ii) Let $C, D, E \in \mathrm{Cl}(G)$, $e \in E$ and $g \in G$. Then the map $(c, d) \mapsto (gcg^{-1}, gdg^{-1})$ is a bijection between $\{(c, d) \in C \times D : cd = e\}$ and $\{(c, d) \in C \times D : cd = geg^{-1}\}$. Therefore, the *class multiplication constant*

$$c_{CDE} := |\{(c, d) \in C \times D : cd = e\}|$$

does not depend on the choice of $e \in E$.

Lemma 1.23. *For an irreducible representation Δ with character χ and $g \in C \in \mathrm{Cl}(G)$, it holds that*

$$\sum_{x \in C} \Delta(x) = \omega_\Delta(C)\,\mathrm{id}$$

with $\omega_\Delta(C) := \omega_\chi(C) := \frac{|C|}{\chi(1)}\chi(g)$.

Proof. Let $A := \sum_{x \in C} \Delta(x)$. For $y \in G$, we have $\Delta(y)A\Delta(y^{-1}) = \sum_{x \in C} \Delta(yxy^{-1}) = A$. From Schur's Lemma, it follows that $A = \omega_\Delta(C)\,\mathrm{id}$ for some $\omega_\Delta(C) \in \mathbb{C}$. Furthermore, $\omega_\Delta(C)\chi(1) = \mathrm{Trace}\, A = \sum_{x \in C} \chi(x) = |C|\chi(g)$. □

Theorem 1.24 (Second orthogonality relation)**.** *For $g, h \in G$, it holds that*

$$\boxed{\sum_{\chi \in \mathrm{Irr}(G)} \chi(g)\overline{\chi(h)} = \begin{cases} |\mathrm{C}_G(g)| & \text{if } g \text{ and } h \text{ are conjugate,} \\ 0 & \text{otherwise.} \end{cases}}$$

Proof. Let $C, D \in \mathrm{Cl}(G)$ with $g \in C$ and $h^{-1} \in D$. Let $\Delta\colon G \to \mathrm{GL}(n, \mathbb{C})$ be an irreducible matrix representation with character χ. According to Lemma 1.23, it holds that

$$\begin{aligned}\omega_\chi(C)\omega_\chi(D)1_n &= \sum_{c\in C}\Delta(c)\sum_{d\in D}\Delta(d) = \sum_{c\in C}\sum_{d\in D}\Delta(cd) \overset{1.22}{=} \sum_{E\in\mathrm{Cl}(G)} c_{CDE}\sum_{e\in E}\Delta(e)\\ &= \sum_{E\in\mathrm{Cl}(G)} c_{CDE}\,\omega_\chi(E)1_n.\end{aligned}$$

From the definition of ω_χ, one obtains

$$\chi(g)\overline{\chi(h)} = \sum_{E\in\mathrm{Cl}(G)} \frac{c_{CDE}|E|}{|C||D|}\chi(1)\chi(e),$$

where in each case $e \in E$ is chosen. Let ρ be the regular character of G. Summing over χ yields

$$\sum_{\chi\in\mathrm{Irr}(G)} \chi(g)\overline{\chi(h)} = \sum_{E\in\mathrm{Cl}(G)} \frac{c_{CDE}|E|}{|C||D|}\sum_{\chi\in\mathrm{Irr}(G)}\chi(1)\chi(e) \overset{1.22}{=} \sum_{E\in\mathrm{Cl}(G)}\frac{c_{CDE}|E|}{|C||D|}\rho(e) = \frac{c_{CD\{1\}}|G|}{|C||D|}.$$

The conjugacy class of h is clearly $D^{-1} = \{d^{-1} : d \in D\}$. If g and h are not conjugate, then $C\cap D^{-1} = \varnothing$ and $c_{CD\{1\}} = 0$. Otherwise, $c_{CD\{1\}} = |C| = |D|$ and the assertion follows from $\frac{|G|}{|C|} = |\mathrm{C}_G(g)|$. ☐

Theorem 1.25. $\mathrm{Irr}(G)$ *is an orthonormal basis of* $\mathrm{CF}(G)$. *In particular,* $\boxed{k(G) := |\mathrm{Irr}(G)| = |\mathrm{Cl}(G)|.}$

Proof. We already know that $\mathrm{Irr}(G)$ is linearly independent (Remark 1.20). According to the second orthogonality relation, for $g \in C \in \mathrm{Cl}(G)$ on the other hand

$$\varphi_C := \frac{1}{|\mathrm{C}_G(g)|}\sum_{\chi\in\mathrm{Irr}(G)}\chi(g^{-1})\chi$$

is the characteristic function on C (i. e. $\varphi_C(x)$ is 1 if $x \in C$ and 0 otherwise). Since the characteristic functions form a basis of $\mathrm{CF}(G)$, $\mathrm{Irr}(G)$ is also a generating set. The orthonormality follows from the first orthogonality relation. ☐

Remark 1.26.

(i) Every class function $f \in \mathrm{CF}(G)$ can thus be uniquely written in the form

$$f = \sum_{\chi\in\mathrm{Irr}(G)} a_\chi\chi$$

with $a_\chi \in \mathbb{C}$. If $a_\chi \in \mathbb{Z}$ for all $\chi \in \mathrm{Irr}(G)$, then f is a *virtual character* of G (or *generalized character*). If additionally $a_\chi \ge 0$ for all $\chi \in \mathrm{Irr}(G)$ and $a_\psi > 0$ for at least one $\psi \in \mathrm{Irr}(G)$, then f is a character according to Remark 1.13(iv). Conversely, every character of G has the form $\psi = \sum_{\chi\in\mathrm{Irr}(G)} a_\chi\chi$ with $a_\chi \in \mathbb{N}_0$. If $a_\chi = (\psi,\chi)_G > 0$, then χ is called an *irreducible constituent* of ψ with *multiplicity* a_χ. Furthermore, $(\psi,\psi)_G = \sum a_\chi^2$ holds. In particular, ψ is irreducible if and only if $(\psi,\psi)_G = 1$ holds.

(ii) In general, no canonical bijection between $\mathrm{Cl}(G)$ and $\mathrm{Irr}(G)$ is known.

2 Character Tables

Remark 2.1. Let $g_1, \dots, g_k \in G$ be a system of representatives for the conjugacy classes of G, and let $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_k\}$. The $k \times k$-matrix $C := (\chi_i(g_j))_{i,j}$ is called the *character table* of G. Of course, C depends on the order of the elements and characters. Usually, one chooses $g_1 = 1$, $\chi_1 = 1_G$ and $\chi_1(1) \le \chi_2(1) \le \ldots \le \chi_k(1)$. In this chapter, we want to calculate C for some groups. The first orthogonality relation can be written in the form

$$\sum_{i=1}^{k} \frac{1}{|\mathrm{C}_G(g_i)|} \chi_r(g_i)\overline{\chi_s(g_i)} = \delta_{rs}.$$

This thus concerns the rows of C. The second orthogonality relation states that the columns of C are pairwise orthogonal with respect to the standard inner product of $\mathbb{C}^k$. In particular, C is invertible.

Remark 2.2. Let G and H be finite groups and $\Delta\colon G \to \mathrm{GL}(n, \mathbb{C})$ and $\Gamma\colon H \to \mathrm{GL}(m, \mathbb{C})$ be matrix representations. For $g \in G$ and $h \in H$ we write $\Delta(g) = (\alpha_{ij}(g))$ and $\Gamma(h) = (\beta_{ij}(h))$. Let $f : \{1, \dots, nm\} \to \{1, \dots, n\} \times \{1, \dots, m\}$, $i \mapsto (i_1, i_2)$ be a bijection. For $(g, h) \in G \times H$ we define a matrix $(\Delta \otimes \Gamma)(g, h) \in \mathbb{C}^{nm \times nm}$ by

$$(\Delta \otimes \Gamma)(g, h) = (\alpha_{i_1 j_1}(g)\beta_{i_2 j_2}(h))_{i,j=1}^{nm} \qquad \text{(Kronecker product).}$$

Theorem 2.3. *The map $\Delta \otimes \Gamma\colon G \times H \to \mathrm{GL}(nm, \mathbb{C})$ is a representation with degree nm. For the corresponding characters, $\chi_{\Delta \otimes \Gamma} = \chi_\Delta \chi_\Gamma$ holds, where $(\chi_\Delta \chi_\Gamma)(g, h) = \chi_\Delta(g)\chi_\Gamma(h)$ for $g \in G$ and $h \in H$.*

Proof. For $(g_1, h_1), (g_2, h_2) \in G \times H$ we have

$$\begin{aligned}
(\Delta \otimes \Gamma)(g_1 g_2, h_1 h_2) &= (\alpha_{i_1 j_1}(g_1 g_2)\beta_{i_2 j_2}(h_1 h_2))_{i,j} \\
&= \left(\sum_{k=1}^{n} \sum_{l=1}^{m} \alpha_{i_1 k}(g_1)\alpha_{k j_1}(g_2)\beta_{i_2 l}(h_1)\beta_{l j_2}(h_2) \right)_{i,j} \\
&= \left(\sum_{k=1}^{n} \sum_{l=1}^{m} \alpha_{i_1 k}(g_1)\beta_{i_2 l}(h_1)\alpha_{k j_1}(g_2)\beta_{l j_2}(h_2) \right)_{i,j} \\
&= \left(\sum_{r=1}^{nm} \alpha_{i_1 r_1}(g_1)\beta_{i_2 r_2}(h_1)\alpha_{r_1 j_1}(g_2)\beta_{r_2 j_2}(h_2) \right)_{i,j} \\
&= (\Delta \otimes \Gamma)(g_1, h_1)(\Delta \otimes \Gamma)(g_2, h_2).
\end{aligned}$$

In particular, $(\Delta \otimes \Gamma)(g, h)(\Delta \otimes \Gamma)(g^{-1}, h^{-1}) = (\Delta \otimes \Gamma)(1, 1) = (\alpha_{i_1 j_1}(1)\beta_{i_2 j_2}(1))_{i,j} = 1_{nm}$ and $(\Delta \otimes \Gamma)(g, h) \in \mathrm{GL}(nm, \mathbb{C})$. This shows that $\Delta \otimes \Gamma$ is a representation. For the character, we have

$$\chi_{\Delta \otimes \Gamma}(g, h) = \sum_{i=1}^{nm} \alpha_{i_1 i_1}(g)\beta_{i_2 i_2}(h) = \sum_{k=1}^{n} \sum_{l=1}^{m} \alpha_{kk}(g)\beta_{ll}(h) = \chi_\Delta(g)\chi_\Gamma(h). \qquad \square$$

Remark 2.4.

(i) According to Theorem 1.21, the similarity class of $\Delta \otimes \Gamma$ does not depend on the choice of the bijection f.

(ii) In the case $H = G$, one obtains a representation of G by $g \mapsto (\Delta \otimes \Gamma)(g, g)$. This is also denoted by $\Delta \otimes \Gamma$. For characters χ, ψ of G, the product $\chi\psi$ with $(\chi\psi)(g) := \chi(g)\psi(g)$ is therefore also a character of G. For $\chi, \psi \in \mathrm{Irr}(G)$, $\chi\psi$ is not necessarily irreducible.

Theorem 2.5. *For finite groups G, H,* $\mathrm{Irr}(G \times H) = \{\chi\psi : \chi \in \mathrm{Irr}(G), \psi \in \mathrm{Irr}(H)\}$.

Proof. Let $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_n\}$ and $\mathrm{Irr}(H) = \{\psi_1, \ldots, \psi_m\}$. Then

$$\begin{aligned}(\chi_i\psi_j, \chi_k\psi_l)_{G\times H} &= \frac{1}{|G \times H|}\sum_{g\in G}\sum_{h\in H}\chi_i(g)\psi_j(h)\overline{\chi_k(g)\psi_l(h)} \\ &= \left(\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_k(g)}\right)\left(\frac{1}{|H|}\sum_{h\in H}\psi_j(h)\overline{\psi_l(h)}\right) = \delta_{ik}\delta_{jl}.\end{aligned}$$

Thus the characters $\chi_i\psi_j$ are irreducible and pairwise distinct. Because of

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(\chi_i\psi_j)(1)^2 = \sum_{i=1}^{n}\chi_i(1)^2\sum_{j=1}^{m}\psi_j(1)^2 = |G||H| = |G \times H|$$

one has found all irreducible characters of $G \times H$. □

Remark 2.6. Let G be cyclic of order n (we write $G \cong C_n$). According to Exercise 1, the character table of G is given by $(e^{\frac{2\pi ikl}{n}})_{k,l=0}^{n-1}$ $(i = \sqrt{-1})$. From Algebra 1/2 it is known that every abelian group G is the direct product of cyclic groups (for an elementary proof see Theorem 2.1.3 in Kurzweil-Stellmacher, "Theorie der endlichen Gruppen"). With Theorem 2.5, the character table of G can thus be easily calculated.

Example 2.7. Let $G := \{1, x, y, z\,(= xy)\} \cong C_2 \times C_2 = C_2^2$ be the Klein four-group with $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_4\}$. Then

C_2^2	1	x	y	z
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	1	-1	-1
χ_4	1	-1	-1	1

is the character table of G.

Definition 2.8. For $x, y \in G$, $[x, y] := xyx^{-1}y^{-1}$ is the *commutator* of x and y. We set

$$G' := \langle [x, y] : x, y \in G\rangle$$

(the smallest subgroup containing all commutators). Then G' is called the *derived subgroup* of G.

Remark 2.9. For $\alpha \in \mathrm{Aut}(G)$ and $x, y \in G$, it is obvious that $\alpha([x, y]) = [\alpha(x), \alpha(y)] \in G'$. In particular, $G' \trianglelefteq G$ (choose $\alpha \in \mathrm{Inn}(G)$). For $xG', yG' \in G/G'$, we have

$$xG'yG' = xy\underbrace{[y^{-1}, x^{-1}]}_{\in G'}G' = yG'xG',$$

i. e. G/G' is abelian. Conversely, if $N \trianglelefteq G$ with G/N abelian, then $[x, y]N = xNyN(xN)^{-1}(yN)^{-1} = N$ for $x, y \in G$, i. e. $G' \subseteq N$.

Theorem 2.10. *The characters of G of degree 1 are precisely the inflations of* $\operatorname{Irr}(G/G')$.

Proof. Let χ be a character of G of degree 1. Then $\chi\colon G \to \mathbb{C}^\times$ is a homomorphism. In particular, $G/\operatorname{Ker}\chi$ is abelian as a subgroup of $\mathbb{C}^\times$, i. e. $G' \subseteq \operatorname{Ker}\chi$. Deflation thus yields a $\psi \in \operatorname{Irr}(G/G')$ and χ is the inflation of ψ.

Conversely, the inflation of every $\chi \in \operatorname{Irr}(G/G')$ has degree 1 because of Theorem 1.10. □

Example 2.11. Let $G = A_4$ be the alternating group of degree 4. As is well known, the Klein four-group $V := \langle (1,2)(3,4), (1,3)(2,4) \rangle$ is normal in G. Because $|G/V| = 3$, G/V is abelian and $G' \subseteq V$. Since G is not abelian, $G' = V$ must hold. For $\operatorname{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$, it thus holds wlog. $\chi_1(1) = \chi_2(1) = \chi_3(1) = 1$ and $\chi_i(1) > 1$ for $i \geq 4$. Furthermore, $12 = |G| = \sum_{i=1}^k \chi_i(1)^2 = 3 + \sum_{i=4}^k \chi_i(1)^2$. It follows that $k = 4$ and $\chi_4(1) = 3$. Thus G also has 4 conjugacy classes. For reasons of order, the elements 1, $(1,2)(3,4)$ and $(1,2,3)$ are pairwise not conjugate. In the abelian group G/G', $(1,2,3)G'$ and $(1,3,2)G' = (1,2,3)^{-1}G'$ are also not conjugate. Thus $(1,2,3)$ and $(1,3,2)$ cannot be conjugate in G either. Therefore 1, $(1,2)(3,4)$, $(1,2,3)$ and $(1,3,2)$ is a system of representatives for the conjugacy classes of G. A part of the character table is now obtained as follows

A_4	1	$(1,2)(3,4)$	$(1,2,3)$	$(1,3,2)$
χ_1	1	1	1	1
χ_2	1	1	σ	σ^{-1}
χ_3	1	1	σ^{-1}	σ
χ_4	3			

$$\sigma := e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{-3}}{2}.$$

The last row results from the second orthogonality relation:

A_4	1	$(1,2)(3,4)$	$(1,2,3)$	$(1,3,2)$
χ_1	1	1	1	1
χ_2	1	1	σ	σ^{-1}
χ_3	1	1	σ^{-1}	σ
χ_4	3	-1	0	0

Lemma 2.12. *Let $g \in G$. For a representation Δ of G with character χ, the following holds*

(i) $|\chi(g)| \leq \chi(1)$.

(ii) $|\chi(g)| = \chi(1) \Leftrightarrow \Delta(g) \in \mathbb{C}^\times \operatorname{id}$.

(iii) $\chi(g) = \chi(1) \Leftrightarrow g \in \operatorname{Ker}(\Delta)$.

Proof. Let $n := \chi(1)$, and let $\epsilon_1, \ldots, \epsilon_n \in \mathbb{C}$ be the eigenvalues of $\Delta(g)$. Because $(\Delta(g))^{|\langle g \rangle|} = \Delta(g^{|\langle g \rangle|}) = \Delta(1) = 1_n$, the ϵ_i are roots of unity. We apply the Cauchy-Schwarz inequality to the vectors $v := (\epsilon_1, \ldots, \epsilon_n)$ and $w := (1, \ldots, 1)$:

$$|\chi(g)| = |\epsilon_1 + \ldots + \epsilon_n| = |\langle v, w \rangle| \leq \|v\|\|w\| = \sqrt{n}\sqrt{n} = n.$$

This shows (i). If equality holds, then v and w are linearly dependent and it follows $\epsilon := \epsilon_1 = \epsilon_2 = \ldots = \epsilon_n$. Since $\Delta(g)$ is diagonalizable (Exercise 4), the geometric multiplicity of the eigenvalue ϵ is equal to n, i. e. $\Delta(g) = \epsilon \operatorname{id}$. Conversely, if $\Delta(g) \in \mathbb{C}^\times \operatorname{id}$, then $|\chi(g)| = \chi(1)$ certainly follows. If even $\chi(g) = \chi(1)$, then obviously $\epsilon = 1$ and $g \in \operatorname{Ker}(\Delta)$. The converse is also clear here. □

Definition 2.13. For a representation Δ with character χ we set $\operatorname{Ker}(\chi) := \operatorname{Ker}(\Delta)$ and $\mathrm{Z}(\chi) := \mathrm{Z}(\Delta) := \{g \in G : |\chi(g)| = \chi(1)\}$. One calls $\mathrm{Z}(\chi)$ the *center* of χ (resp. Δ).

Theorem 2.14. *For every character χ of G, $\operatorname{Ker}(\chi)$ and $\mathrm{Z}(\chi)$ are normal subgroups of G. Here $\operatorname{Ker}(\chi) \le \mathrm{Z}(\chi)$ and $\mathrm{Z}(\chi)/\operatorname{Ker}(\chi)$ is cyclic. If $\chi \in \operatorname{Irr}(G)$, then $\mathrm{Z}(\chi)/\operatorname{Ker}(\chi) = \mathrm{Z}(G/\operatorname{Ker}(\chi))$ and $\mathrm{Z}(G) \subseteq \mathrm{Z}(\chi)$.*

Proof. Certainly $\operatorname{Ker}(\chi) \trianglelefteq G$ and $\operatorname{Ker}(\chi) \subseteq \mathrm{Z}(\chi)$. Let $\Delta\colon G \to \mathrm{GL}(V)$ be a representation with character χ. Obviously $\mathbb{C}^\times \operatorname{id}_V \subseteq \mathrm{Z}(\mathrm{GL}(V))$ and thus $\mathbb{C}^\times \operatorname{id}_V \trianglelefteq \mathrm{GL}(V)$. Consequently $\mathrm{Z}(\chi) = \Delta^{-1}(\mathbb{C}^\times \operatorname{id}_V) \trianglelefteq G$ as well. According to the isomorphism theorem, $\mathrm{Z}(\chi)/\operatorname{Ker}(\chi)$ is furthermore isomorphic to a finite subgroup H of $\mathbb{C}^\times \operatorname{id}_V \cong \mathbb{C}^\times$. Obviously H consists exactly of the $|H|$-th roots of unity in $\mathbb{C}$. In particular, H is cyclic (in Algebra 1 one proves this for arbitrary fields).

Now let $\chi \in \operatorname{Irr}(G)$. By deflation we can assume $\operatorname{Ker}(\chi) = 1$ and $G \le \mathrm{GL}(V)$ (this does not change $\mathrm{Z}(\chi)$). Obviously then

$$\mathrm{Z}(\chi) \subseteq \mathbb{C}^\times \operatorname{id}_V \cap G \subseteq \mathrm{Z}(\mathrm{GL}(V)) \cap G \subseteq \mathrm{Z}(G).$$

For $x \in \mathrm{Z}(G)$ conversely $\Delta(g)\Delta(x) = \Delta(gx) = \Delta(xg) = \Delta(x)\Delta(g)$ holds for all $g \in G$. Schur's Lemma shows $\Delta(x) \in \mathbb{C}^\times \operatorname{id}_V$ and thus $x \in \mathrm{Z}(\chi)$.

The last statement follows from $\mathrm{Z}(G)\operatorname{Ker}(\chi)/\operatorname{Ker}(\chi) \le \mathrm{Z}(G/\operatorname{Ker}(\chi))$. □

Remark 2.15. In this way one can often construct normal subgroups, because every normal subgroup is the kernel of a character (Exercise 9).

Theorem 2.16. *The character table of G can be calculated from the class multiplication constants.*

Proof (Burnside *algorithm*). Let $\mathrm{Cl}(G) = \{K_1, \dots, K_n\}$ and $c_{ijk} := C_{K_iK_jK_k}$ be the class multiplication constant. Let $T_i := (c_{ijk})_{j,k} \in \mathbb{Z}^{n\times n}$. Let $\Delta_1, \dots, \Delta_n$ be the irreducible representations of G and $\omega_i := \omega_{\Delta_i}$ for $i = 1, \dots, n$. According to Lemma 1.23, it holds that

$$\omega_l(K_i)\omega_l(K_j)\operatorname{id} = \sum_{x\in K_i}\Delta_l(x)\sum_{y\in K_j}\Delta_l(y) = \sum_{(x,y)\in K_i\times K_j}\Delta_l(xy) = \sum_{k=1}^n c_{ijk}\omega_l(K_k)\operatorname{id}$$

for $1 \le i, j, l \le n$. Consequently, $e_l := (\omega_l(K_k))_k \in \mathbb{C}^n$ is an eigenvector of T_i with eigenvalue $\omega_l(K_i)$. [1] Clearly, $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{C}^n$. Every eigenspace of T_i is therefore spanned by some of the e_l. We intersect these eigenspaces with the eigenspaces of the T_j for $j \ne i$. The non-trivial intersections have the form

$$V_l := \{v \in \mathbb{C}^n : \forall i : T_iv = \omega_l(K_i)v\} \le \mathbb{C}^n$$

for some $1 \le l \le n$. Since for $l \ne k$ there always exists an i with $\omega_l(K_i) \ne \omega_k(K_i)$, the sum of the V_l is direct. For dimension reasons, it follows that $V_l = \langle e_l\rangle$ for $l = 1, \dots, n$. Because of $\omega_l(K_1) = 1$, e_l can be calculated from V_l. According to the first orthogonality relation, there exists only one vector e_l, say e_1, which consists only of positive numbers. It belongs to the trivial representation Δ_1. From this, the class lengths $|K_i| = \omega_1(K_i)$ for $i = 1, \dots, n$ are obtained. Because of

$$\sum_{i=1}^n \frac{|\omega_l(K_i)|^2}{|K_i|} = \frac{1}{\chi_l(1)^2}\sum_{g\in G}|\chi_l(g)|^2 = \frac{|G|}{\chi_l(1)^2}(\chi_l, \chi_l)_G = \frac{|G|}{\chi_l(1)^2}$$

one obtains $\chi_l(1)$ and subsequently also $\chi_l(g) = \frac{\chi_l(1)\omega_l(K_i)}{|K_i|}$ for $g \in K_i$. □

[1] The eigenvalues of T_i can indeed be calculated, but their assignment to ω_l is not unique.

Remark 2.17. As a rule, one does not need all matrices T_i to calculate the character table. If, for example, $\omega_l(K_i)$ as an eigenvalue of T_i has multiplicity 1, then e_l can be determined directly as a generator of the eigenspace. Optimizations of this kind lead to the *Dixon-Schneider algorithm*, which is frequently used in practice.

Theorem 2.18. *Let* $\mathrm{Cl}(G) = \{K_1, \ldots, K_n\}$ *and* $g_i \in K_i$*. Then*

$$c_{ijk} = \frac{|K_i||K_j|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g_i)\chi(g_j)\overline{\chi(g_k)}}{\chi(1)}$$

for $1 \leq i, j, k \leq n$*. The class multiplication constants can thus be determined from the character table.*

Proof. As in the proof of Theorem 2.16, $\omega_\chi(K_i)\omega_\chi(K_j) = \sum_{k=1}^n c_{ijk}\omega_\chi(K_k)$. From this it follows that

$$\frac{|K_i||K_j|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g_i)\chi(g_j)\overline{\chi(g_k)}}{\chi(1)} = \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \omega_\chi(K_i)\omega_\chi(K_j)\chi(1)\overline{\chi(g_k)}$$
$$= \frac{1}{|G|}\sum_{l=1}^n c_{ijl} \sum_{\chi \in \mathrm{Irr}(G)} \omega_\chi(K_l)\chi(1)\overline{\chi(g_k)} = \frac{1}{|G|}\sum_{l=1}^n c_{ijl}|K_l| \sum_{\chi \in \mathrm{Irr}(G)} \chi(g_l)\overline{\chi(g_k)} \overset{1.24}{=} c_{ijk}.$$ ∎

3 Algebraic integers

Definition 3.1. A number $\zeta \in \mathbb{C}$ is called *algebraic integer*, if it is a root of a monic, integral polynomial, i. e. there exist numbers $n \in \mathbb{N}$ and $a_0, \ldots, a_{n-1} \in \mathbb{Z}$ with $\zeta^n + a_{n-1}\zeta^{n-1} + \ldots + a_1\zeta + a_0 = 0$.

Example 3.2.

(i) Integers are obviously algebraic integers.

(ii) Roots of unity are algebraic integers as roots of polynomials of the form $X^n - 1$.

Lemma 3.3. *If* $\alpha, \beta \in \mathbb{C}$ *are algebraic integers, then so are* $\alpha + \beta$ *and* $\alpha\beta$*. (The algebraic integers thus form a ring.)*

Proof. We write

$$\begin{aligned} \alpha^n &= a_{n-1}\alpha^{n-1} + \ldots + a_0, \\ \beta^m &= b_{m-1}\beta^{m-1} + \ldots + b_0 \end{aligned} \tag{3.1}$$

with $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1} \in \mathbb{Z}$. Let $S := \{\alpha^i\beta^j : i = 0, \ldots, n-1,\ j = 0, \ldots, m-1\}$ and $\gamma := \alpha + \beta$ (resp. $\alpha\beta$). For $s \in S$ there then exist numbers $c_{st} \in \mathbb{Z}$ with $\gamma s = \sum_{t \in S} c_{st}t$ (use (3.1)). For $A := (c_{st})_{s,t \in S} \in \mathbb{Z}^{nm \times nm}$ and $v := (s : s \in S)$ we have $Av = \gamma v$. Thus γ is a root of the monic, integral polynomial $\det(X1_{nm} - A)$. ∎

Remark 3.4. If χ is a character of G, then $\chi(g)$ is an algebraic integer for $g \in G$ as a sum of roots of unity (see for example the proof of Lemma 2.12).

Lemma 3.5. *If* $\zeta \in \mathbb{Q}$ *is an algebraic integer, then* $\zeta \in \mathbb{Z}$*.*

Proof. Let $\zeta = \frac{r}{s}$ with $r, s \in \mathbb{Z}$ and $\gcd(r, s) = 1$. By assumption there exist $a_0, \ldots, a_{n-1} \in \mathbb{Z}$ with

$$\frac{r^n}{s^n} = \frac{a_{n-1}r^{n-1}}{s^{n-1}} + \ldots + \frac{a_1 r}{s} + a_0.$$

Rearranging yields

$$r^n = s(a_{n-1}r^{n-1} + \ldots + a_1 r s^{n-2} + a_0 s^{n-1}).$$

Thus $s \mid r^n$. Because of $\gcd(r, s) = 1$ it follows that $s = \pm 1$ and $\zeta \in \mathbb{Z}$. □

Lemma 3.6. *For $C \in \mathrm{Cl}(G)$ and $\chi \in \mathrm{Irr}(G)$, $\omega_\chi(C)$ is algebraic-integral.*

Proof. As shown in the proof of Theorem 2.16, $\omega_\chi(C)$ is an eigenvalue of an integer matrix T. Thus, $\omega_\chi(C)$ is algebraic-integral as a root of the monic, integer characteristic polynomial of T. □

Theorem 3.7. *For $\chi \in \mathrm{Irr}(G)$, we have* $\boxed{\chi(1) \mid |G|.}$

Proof. Let $g_1, \ldots, g_k \in G$ be representatives for the conjugacy classes $C_1, \ldots, C_k$ of G. According to the first orthogonality relation, it then follows that

$$\frac{|G|}{\chi(1)} = \frac{1}{\chi(1)} \sum_{x \in G} \chi(x)\overline{\chi(x)} = \frac{1}{\chi(1)} \sum_{i=1}^{k} |C_i| \chi(g_i)\chi(g_i^{-1}) = \sum_{i=1}^{k} \omega_\chi(C_i)\chi(g_i^{-1}).$$

By Lemma 3.6, $\frac{|G|}{\chi(1)}$ is algebraic-integral. The assertion now follows from Lemma 3.5. □

Theorem 3.8. *Let $\chi \in \mathrm{Irr}(G)$ and $g \in C \in \mathrm{Cl}(G)$ with $\gcd(\chi(1), |C|) = 1$. Then $g \in \mathrm{Z}(\chi)$ or $\chi(g) = 0$.*

Proof. Let $\alpha := \frac{\chi(g)}{\chi(1)}$. Because of $\gcd(\chi(1), |C|) = 1$, there exist $a, b \in \mathbb{Z}$ with $a\chi(1) + b|C| = 1$. With $\omega_\chi(C)$ and $\chi(g)$, also

$$\alpha = \frac{\chi(g)}{\chi(1)}(a\chi(1) + b|C|) = a\chi(g) + b\omega_\chi(C)$$

is algebraic-integral. Let $n := |\langle g \rangle|$ and $\zeta := e^{\frac{2\pi i}{n}} \in \mathbb{C}$. As a sum of n-th roots of unity, $\chi(g) \in \mathbb{Q}(\zeta)$. Let $\mathcal{G}$ be the Galois group of the Galois extension $\mathbb{Q}(\zeta)|\mathbb{Q}$. For $\sigma \in \mathcal{G}$, $\sigma(\alpha)$ is also algebraic-integral, because α and $\sigma(\alpha)$ are roots of the same integer polynomial. Therefore, $\beta := \prod_{\sigma \in \mathcal{G}} \sigma(\alpha)$ is also algebraic-integral. Because of $\sigma(\beta) = \beta$ for all $\sigma \in \mathcal{G}$, β lies in the fixed field of $\mathcal{G}$, i.e., $\beta \in \mathbb{Q}$ (Galois theory). By Lemma 3.5, $\beta \in \mathbb{Z}$. In the case $g \notin \mathrm{Z}(\chi)$, we have $|\alpha| < 1$ (Lemma 2.12). With $\chi(g)$, also $\sigma(\chi(g))$ is a sum of $m := \chi(1)$ many n-th roots of unity $\epsilon_1, \ldots, \epsilon_m$. It follows that

$$|\sigma(\chi(g))| = |\epsilon_1 + \ldots + \epsilon_m| \le |\epsilon_1| + \ldots + |\epsilon_m| = m$$

and $|\sigma(\alpha)| \le 1$ for $\sigma \in \mathcal{G}$. Consequently, $|\beta| < 1$, i.e., $\beta = 0$. Thus $\alpha = 0$ and $\chi(g) = 0$. □

Theorem 3.9. *Let G be simple and non-abelian, $C \in \mathrm{Cl}(G)$ and $|C|$ a power of a prime p. Then $C = \{1\}$.*

Proof. We assume $C \neq \{1\}$ and choose $g \in C$ and $\chi \in \mathrm{Irr}(G) \setminus \{1_G\}$. Since G is simple, $\mathrm{Ker}(\chi) = 1$. Since G is non-abelian, $\mathrm{Z}(\chi) = 1$ as well (Theorem 2.14). In the case $p \nmid \chi(1)$, it follows that $\chi(g) = 0$ according to Theorem 3.8. Therefore,

$$\sum_{\substack{\chi\in\mathrm{Irr}(G),\\ p\mid\chi(1)}} \frac{\chi(1)}{p}\chi(g) = \frac{1}{p}\sum_{1_G\neq\chi\in\mathrm{Irr}(G)} \chi(1)\chi(g) = \frac{1}{p}\Bigl(\sum_{\chi\in\mathrm{Irr}(G)} \chi(1)\chi(g) - 1_G(1)1_G(g)\Bigr) = -\frac{1}{p} \in \mathbb{Q}\setminus\mathbb{Z}$$

is algebraic-integral. Contradiction. □

Theorem 3.10 (Burnside)**.** *Let $|G| = p^a q^b$ with prime numbers p, q and $a, b \in \mathbb{N}_0$. Then G is solvable.*

Proof. (Induction on $|G|$) wlog. let $G \neq 1$. Let N be a maximal normal subgroup of G. If $N \neq 1$, then N and G/N are solvable by induction, and thus so is G. Therefore, let $N = 1$, i.e., G is simple and wlog. non-abelian. Let $P \neq 1$ be a Sylow subgroup of G, $g \in \mathrm{Z}(P) \setminus \{1\}$ and C the conjugacy class of g. Then $|C| = |G : \mathrm{C}_G(g)| \bigm| |G : P| = q^b$ is a prime power. According to Theorem 3.9, $C = \{1\}$. Contradiction. □

Theorem 3.11. *For $\chi \in \mathrm{Irr}(G)$,* $\boxed{\chi(1) \bigm| |G : \mathrm{Z}(\chi)|.}$

Proof (Navarro). Let $\Delta\colon G \to \mathrm{GL}(V)$ be a representation with character χ. Because $|G : \mathrm{Z}(\chi)| = |G/\mathrm{Ker}(\chi) : \mathrm{Z}(\chi)/\mathrm{Ker}(\chi)|$, we can replace Δ by its deflation $G/\mathrm{Ker}(\chi) \to \mathrm{GL}(V)$ and assume $\mathrm{Ker}(\chi) = 1$. According to Theorem 2.14, $Z := \mathrm{Z}(\chi) = \mathrm{Z}(G)$ and $\Delta(z) = \lambda(z)\,\mathrm{id}_V$ with $\lambda(z) \in \mathbb{C}$ for $z \in Z$. Let $K_1, \ldots, K_s \in \mathrm{Cl}(G)$ be the conjugacy classes on which χ does not vanish. For $g_i \in K_i$ and $z \in Z$, we have $\Delta(g_i z) = \Delta(g_i)\lambda(z)$ and

$$\chi(g_i z) = \chi(g_i)\lambda(z) = \frac{\chi(g_i)\chi(z)}{\chi(1)}.$$

Since χ is faithful, it follows that $\chi(g_i z) \neq \chi(g_i)$ for $z \neq 1$. Possibly, g_i and $g_i z$ are not conjugate. This shows $K_i z = K_j$ for some $j \neq i$. With a suitable ordering, we now have

$$\bigcup_{i=1}^{s} K_i = \bigcup_{i=1}^{t} \dot{\bigcup_{z\in Z}} K_i z$$

with $t|Z| = s$. Furthermore,

$$\chi(g_i z)\overline{\chi(g_i z)} = \chi(g_i)\overline{\chi(g_i)}\frac{|\chi(z)|^2}{\chi(1)^2} = \chi(g_i)\overline{\chi(g_i)}.$$

Therefore,

$$\frac{|G : Z|}{\chi(1)} = \sum_{g\in G} \frac{\chi(g)\overline{\chi(g)}}{|Z|\chi(1)} = \sum_{i=1}^{t}\sum_{z\in Z} \frac{|K_i|\chi(g_i z)\overline{\chi(g_i z)}}{|Z|\chi(1)} = \sum_{i=1}^{t} \frac{|K_i|\chi(g_i)\overline{\chi(g_i)}}{\chi(1)} = \sum_{i=1}^{t} \omega_\chi(K_i)\overline{\chi(g_i)}$$

is algebraic-integral according to Lemma 3.6. The assertion follows from Lemma 3.5. □

4 Clifford Theory

Remark 4.1. For $H \leq G$ and $\varphi \in \mathrm{CF}(G)$, the restriction $\varphi_H \in \mathrm{CF}(H)$ is obviously a class function. Conversely, we will now construct a class function on G from $\varphi \in \mathrm{CF}(H)$.

Definition 4.2. For $H \leq G$ and $\varphi \in \mathrm{CF}(H)$, let

$$\boxed{\varphi^G : G \to \mathbb{C},\ x \mapsto \frac{1}{|H|} \sum_{\substack{g \in G,\\ gxg^{-1} \in H}} \varphi(gxg^{-1}).}$$

One calls φ^G the *induction* of φ.

Theorem 4.3. *For $\varphi \in \mathrm{CF}(H)$, we have $\varphi^G \in \mathrm{CF}(G)$.*

Proof. For $x, y \in G$, we have

$$\varphi^G(yxy^{-1}) = \frac{1}{|H|} \sum_{\substack{g \in G,\\ gyxy^{-1}g^{-1} \in H}} \varphi(gyxy^{-1}g^{-1}) = \frac{1}{|H|} \sum_{\substack{h \in G,\\ hxh^{-1} \in H}} \varphi(hxh^{-1}) = \varphi^G(x). \qquad \square$$

Remark 4.4.

(i) It is easy to see that induction is a linear map from $\mathrm{CF}(H)$ to $\mathrm{CF}(G)$.

(ii) For $H \leq G$, $\varphi \in \mathrm{CF}(H)$ and $x \in G$, we have

$$\begin{aligned}\varphi^G(x) &= \frac{1}{|H|} \sum_{\substack{g \in G,\\ g^{-1}xg \in H}} \varphi(g^{-1}xg) = \frac{1}{|H|} \sum_{gH \in G/H} \sum_{\substack{h \in H,\\ h^{-1}g^{-1}xgh \in H}} \varphi(h^{-1}g^{-1}xgh)\\ &= \frac{1}{|H|} \sum_{\substack{gH \in G/H, h \in H,\\ g^{-1}xg \in H}} \varphi(g^{-1}xg) = \sum_{\substack{gH \in G/H,\\ xgH = gH}} \varphi(g^{-1}xg).\end{aligned}$$

This is useful for practical calculation.

Theorem 4.5.

(i) For $K \leq H \leq G$ and $\varphi \in \mathrm{CF}(K)$, we have $(\varphi^H)^G = \varphi^G$. Thus, the induction of class functions is transitive.

(ii) For $\chi \in \mathrm{CF}(G)$ and $\varphi \in \mathrm{CF}(H)$, we have $\boxed{\chi\varphi^G = (\chi_H\varphi)^G}$ *and* $\boxed{(\chi, \varphi^G)_G = (\chi_H, \varphi)_H}$ (Frobenius reciprocity).

Proof.

(i) By Remark 4.4(ii), we have

$$\begin{aligned}(\varphi^H)^G(x) &= \sum_{\substack{gH \in G/H,\\ xgH = gH}} \varphi^H(g^{-1}xg) = \sum_{\substack{gH \in G/H,\\ xgH = gH}} \sum_{\substack{hK \in H/K,\\ g^{-1}xghK = hK}} \varphi(h^{-1}g^{-1}xgh)\\ &= \sum_{\substack{aK \in G/K,\\ xaK = aK}} \varphi(a^{-1}xa) = \varphi^G(x)\end{aligned}$$

for $x \in G$.

(ii) As in (i), we have

$$(\chi\varphi^G)(x) = \chi(x) \sum_{\substack{gH \in G/H,\\ xgH = gH}} \varphi(g^{-1}xg) = \sum_{\substack{gH \in G/H,\\ xgH = gH}} (\chi\varphi)(g^{-1}xg) = (\chi_H\varphi)^G(x)$$

for $x \in G$ and

$$\begin{aligned}(\chi, \varphi^G)_G &= \frac{1}{|G|} \sum_{x \in G} \chi(x) \sum_{\substack{gH \in G/H,\\ xgH = gH}} \overline{\varphi(g^{-1}xg)} = \frac{1}{|G|} \sum_{gH \in G/H} \sum_{\substack{x \in G,\\ g^{-1}xg \in H}} \chi(g^{-1}xg)\overline{\varphi(g^{-1}xg)} \\ &= \frac{1}{|G|} \sum_{gH \in G/H} \sum_{h \in H} \chi(h)\overline{\varphi(h)} = \frac{|G/H|}{|G|} \sum_{h \in H} \chi(h)\overline{\varphi(h)} = (\chi_H, \varphi)_H. \end{aligned}$$

□

Remark 4.6. Frobenius reciprocity states that restriction and induction are adjoint maps between $\mathrm{CF}(G)$ and $\mathrm{CF}(H)$.

Theorem 4.7. *For a character φ of $H \le G$, φ^G is a character of G of degree $|G : H|\varphi(1)$.*

Proof. We write $\varphi^G = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi$ with $a_\chi \in \mathbb{C}$. Then

$$a_\chi = (\chi, \varphi^G)_G = (\chi_H, \varphi)_H \in \mathbb{N}_0,$$

because χ_H is a character of H. Thus φ^G is a character of G. Clearly, $\varphi^G(1) = |G : H|\varphi(1)$ also holds. □

Example 4.8.

(i) 1_1^G is the regular character of G. In particular, φ^G is not necessarily irreducible if φ is irreducible. If φ is reducible, then φ^G must also be reducible due to the linearity of induction.

(ii) Let $H := \langle(1, 2, 3)\rangle$ and $G := S_3$. Let φ be a non-trivial character of H of degree 1. Then $\varphi^G(1) = 2$, $\varphi^G((1, 2)) = 0$, $\varphi^G((1, 2, 3)) = -1$. In particular, $\varphi^G \in \mathrm{Irr}(G)$.

Definition 4.9. Let $H \le G$, $\varphi \in \mathrm{CF}(H)$ and $g \in G$. Then ${}^g\varphi \in \mathrm{CF}(gHg^{-1})$ with ${}^g\varphi(x) := \varphi(g^{-1}xg)$ for $x \in gHg^{-1}$. We call $G_\varphi := \{g \in G : {}^g\varphi = \varphi\} \le G$ the *inertia group* of φ. Furthermore, let

$$\mathrm{Irr}(G|\varphi) := \{\chi \in \mathrm{Irr}(G) : (\chi_H, \varphi)_H \ne 0\}.$$

Remark 4.10.

(i) Clearly $H \le G_\varphi \le \mathrm{N}_G(H) := \{g \in G : gHg^{-1} = H\}$ ($\mathrm{N}_G(H)$ is the *normalizer* of H in G).

(ii) As usual,

$${}^g\varphi = {}^h\varphi \iff h^{-1}g \in G_\varphi \iff gG_\varphi = hG_\varphi$$

for $g, h \in G$.

(iii) Let $\Delta\colon H \to \mathrm{GL}(V)$ be a representation with character χ. For $g \in G$, then clearly ${}^g\Delta\colon gHg^{-1} \to \mathrm{GL}(V)$, $x \mapsto \Delta(g^{-1}xg)$ is a representation of gHg^{-1} with character ${}^g\chi$.

(iv) For $\varphi, \psi \in \mathrm{CF}(H)$ and $g \in G$, we have $\boxed{({}^g\varphi, {}^g\psi)_{gHg^{-1}} = (\varphi, \psi)_H,}$ because

$$\frac{1}{|gHg^{-1}|} \sum_{x \in gHg^{-1}} {}^g\varphi(x)\overline{{}^g\psi(x)} = \frac{1}{|H|} \sum_{x \in H} \varphi(x)\overline{\psi(x)}.$$

(v) For $K \leq H \leq G$, $\varphi \in \mathrm{CF}(H)$ and $g \in G$, we have $\boxed{{}^g(\varphi_K) = ({}^g\varphi)_{gKg^{-1}}.}$

(vi) For $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(N)$, we have $({}^g\chi, {}^g\chi)_N = (\chi, \chi)_N = 1$ and ${}^g\chi \in \mathrm{Irr}(N)$. We say that χ and ${}^g\chi$ are *conjugate*.

Theorem 4.11. *Let $N \trianglelefteq G$, $\chi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{Irr}(N)$ with $e := (\chi_N, \psi)_N \neq 0$. Then*

$$\chi_N = e \sum_{gG_\psi \in G/G_\psi} {}^g\psi.$$

Proof. By Frobenius reciprocity, χ is a constituent of ψ^G. Therefore, χ_N is a constituent of $(\psi^G)_N$. For $x \in N$, it holds that

$$\psi^G(x) = \sum_{\substack{gN \in G/N, \\ xgN = gN}} \psi(gxg^{-1}) = \sum_{gN \in G/N} {}^g\psi(x)$$

according to Remark 4.4. Thus, every irreducible constituent of χ_N is conjugate to ψ. For $g \in G$, it holds that

$$(\chi_N, {}^g\psi)_N = ({}^{g^{-1}}(\chi_N), \psi)_N = (({}^{g^{-1}}\chi)_N, \psi)_N = (\chi_N, \psi)_N = e$$

according to Remark 4.10. This yields the claim. □

Definition 4.12. In the situation of Theorem 4.11, e is called the *ramification index* of χ w.r.t. N. It can be shown that $e \mid |G : N|$ holds (without proof).

Theorem 4.13 (Clifford correspondence)**.** *For $N \trianglelefteq G$ and $\psi \in \mathrm{Irr}(N)$, the map*

$$\Phi : \mathrm{Irr}(G_\psi|\psi) \to \mathrm{Irr}(G|\psi), \quad \chi \mapsto \chi^G$$

is a bijection with $(\chi_N, \psi)_N = ((\chi^G)_N, \psi)_N$.

Proof. Let $\chi \in \mathrm{Irr}(G_\psi|\psi)$ and let φ be an irreducible constituent of χ^G. We first show $\varphi \in \mathrm{Irr}(G|\psi)$. We write $\varphi_{G_\psi} := \sum_{\tau \in \mathrm{Irr}(G_\psi)} a_\tau \tau$ with $a_\tau \geq 0$ and $a_\chi = (\varphi_{G_\psi}, \chi)_{G_\psi} = (\varphi, \chi^G)_G \geq 1$. According to Theorem 4.11, $\chi_N = e\psi$ with $e := (\chi_N, \psi)_N$. It follows that

$$f := (\varphi_N, \psi)_N = \sum_{\tau \in \mathrm{Irr}(G_\psi)} a_\tau (\tau_N, \psi)_N \geq a_\chi (\chi_N, \psi)_N \geq e > 0,$$

i. e. $\varphi \in \mathrm{Irr}(G|\psi)$. Theorem 4.11 implies

$$\varphi(1) = \varphi_N(1) = f \sum_{gG_\psi \in G/G_\psi} {}^g\psi(1) \geq e|G : G_\psi|\psi(1) = |G : G_\psi|\chi(1) = \chi^G(1) \geq \varphi(1).$$

This shows $\chi^G = \varphi \in \mathrm{Irr}(G|\psi)$ and $e = f$. Thus Φ is well-defined.

Now let $\theta \in \mathrm{Irr}(G|\psi)$ be given. Because $\theta_N = (\theta_{G_\psi})_N$, there exists a $\chi \in \mathrm{Irr}(G_\psi|\psi)$ with $(\theta, \chi^G)_G = (\theta_T, \chi)_{G_\psi} \neq 0$. According to the first part of the proof, $\chi^G = \theta$, i.e. Φ is surjective. Furthermore, $(\theta_N, \psi)_N = (\chi_N, \psi)_N$, i.e. χ is the only irreducible constituent of θ_{G_ψ} that lies in $\mathrm{Irr}(G_\psi|\psi)$. This shows the injectivity of Φ. □

Theorem 4.14 (Itô)**.** *Let A be an abelian subgroup of G and $\chi \in \mathrm{Irr}(G)$. Then:*

(i) $\chi(1) \le |G : A|$.

(ii) *If $A \trianglelefteq G$, then* $\chi(1) \mid |G : A|$.

Proof.

(i) Exercise 12.

(ii) Let ψ be an irreducible constituent of χ_A. By Theorem 4.13 there exists a $\widetilde{\chi} \in \mathrm{Irr}(G_\psi)$ with $\widetilde{\chi}^G = \chi$ and $\widetilde{\chi}_A = e\psi$ for some $e \in \mathbb{N}$. Since A is abelian, $|\widetilde{\chi}(x)| = |e\psi(x)| = e\psi(1) = e = \widetilde{\chi}(1)$ for $x \in A$, i.e. $A \subseteq \mathrm{Z}(\widetilde{\chi})$. By Theorem 3.11, $\widetilde{\chi}(1) \mid |G_\psi : \mathrm{Z}(\widetilde{\chi})| \mid |G_\psi : A|$ and therefore $\chi(1) = |G : G_\psi|\widetilde{\chi}(1) \mid |G : G_\psi||G_\psi : A| = |G : A|$. □

Theorem 4.15. *Let $N \trianglelefteq G$ and $\psi \in \mathrm{Irr}(N)$ with $G_\psi = G$. Then $\psi^G = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi \chi$, where e_χ is the ramification index of χ (if $e_\chi > 0$). In particular, $\sum e_\chi^2 = |G : N|$.*

Proof. We write $\psi^G = \sum_{\chi \in \mathrm{Irr}(G)} f_\chi \chi$. In the case $f_\chi \neq 0$, $\chi_N = e_\chi \psi$ by Theorem 4.11. Here $f_\chi = (\chi, \psi^G)_G = (\chi_N, \psi)_N = (e_\chi \psi, \psi)_N = e_\chi$. It follows that

$$|G : N|\psi(1) = \psi^G(1) = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi \chi(1) = \psi(1) \sum_{\chi \in \mathrm{Irr}(G)} e_\chi^2$$

and $\sum e_\chi^2 = |G : N|$. □

Remark 4.16. The numbers e_χ thus behave like character degrees. In the following theorem we will see that they are indeed character degrees if $e_\chi = 1$ holds for some $\chi \in \mathrm{Irr}(G)$.

Theorem 4.17 (Gallagher)**.** *Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$ with $\psi := \chi_N \in \mathrm{Irr}(N)$. Then $\{\lambda\chi : \lambda \in \mathrm{Irr}(G/N)\}$ is the set of irreducible constituents of ψ^G.*

Proof. As usual, we view the characters of G/N as characters of G via inflation. By Exercise 11, $(\chi_N)^G = \chi\rho$, where ρ is the regular character of G/N. Thus

$$\psi^G = \sum_{\lambda \in \mathrm{Irr}(G/N)} \lambda(1)\chi\lambda.$$

By Theorem 4.15,

$$|G : N| = (\psi^G, \psi^G)_G = \sum_{\lambda, \lambda' \in \mathrm{Irr}(G/N)} \lambda(1)\lambda'(1)(\chi\lambda, \chi\lambda')_G \ge \sum_{\lambda \in \mathrm{Irr}(G/N)} \lambda(1)^2 = |G : N|.$$

This shows that the $\chi\lambda$ with $\lambda \in \mathrm{Irr}(G/N)$ are irreducible and pairwise distinct. □

Theorem 4.18. *Let $N \trianglelefteq G$ and G/N cyclic. For every $\psi \in \operatorname{Irr}(N)$ with $G_\psi = G$ there exists a extension $\chi \in \operatorname{Irr}(G)$ with $\chi_N = \psi$.*

Proof. Let $\Delta\colon N \to \mathrm{GL}(n, \mathbb{C})$ be a representation with character ψ. Let $G/N = \langle gN \rangle$ and $k := |G/N|$. Because of $G_\psi = G$, Δ and ${}^g\Delta$ are similar. Thus, let $A \in \mathrm{GL}(n, \mathbb{C})$ with $A\Delta(x)A^{-1} = \Delta(gxg^{-1})$ for all $x \in N$. Inductively, it follows that

$$A^k\Delta(x)A^{-k} = \Delta(g^kxg^{-k}) = \Delta(g^k)\Delta(x)\Delta(g^k)^{-1}$$

for all $x \in N$. By Schur's Lemma, $A^{-k}\Delta(g^k) = \lambda 1_n$ for some $\lambda \in \mathbb{C}^\times$. Let $\mu \in \mathbb{C}^\times$ with $\mu^k = \lambda$. Because of $(\mu A)^k = \lambda A^k = \Delta(g^k)$, the map

$$\Gamma\colon G \to \mathrm{GL}(n, \mathbb{C}), \qquad g^ix \mapsto (\mu A)^i\Delta(x)$$

with $i \in \mathbb{Z}$ and $x \in N$ is well-defined. For $i, j \in \mathbb{Z}$ and $x, y \in N$, it holds that

$$\begin{aligned}\Gamma(g^ix \cdot g^jy) &= \Gamma(g^{i+j} \cdot (g^{-j}xg^j)y) = (\mu A)^{i+j}\Delta(g^{-j}xg^j)\Delta(y) = (\mu A)^{i+j}A^{-j}\Delta(x)A^j\Delta(y)\\ &= (\mu A)^i\Delta(x)(\mu A)^j\Delta(y) = \Gamma(g^ix)\Gamma(g^jy).\end{aligned}$$

Thus, Γ is a representation that extends Δ. Since Δ is irreducible, Γ must also be irreducible. This shows the claim. □

Example 4.19. We calculate the character table of $G := S_4$ from the character table of $N := A_4$. The characters χ_2 and χ_3 of N constructed in Example 2.11 are conjugate under G. Therefore, $\chi_2^G = \chi_3^G \in \operatorname{Irr}(G)$. Since G/N is cyclic, the remaining characters each have two extensions to G. This already yields the following part of the character table:

S_4	1	$(1,2)$	$(1,2)(3,4)$	$(1,2,3)$	$(1,2,3,4)$
1_G	1	1	1	1	1
sgn	1	-1	1	1	-1
χ_2^G	2	0	2	-1	0
ψ	3	a	-1	0	b
$\psi\,\mathrm{sgn}$	3	$-a$	-1	0	$-b$

From the second orthogonality relation, it follows that $ab = -1$. According to Lemma 2.12, a is a sum of second roots of unity and thus an integer. Since b is algebraic-integral, it follows that $a = -b = \pm 1$.

S_4	1	$(1,2)$	$(1,2)(3,4)$	$(1,2,3)$	$(1,2,3,4)$
1_G	1	1	1	1	1
sgn	1	-1	1	1	-1
χ_2^G	2	0	2	-1	0
ψ	3	1	-1	0	-1
$\psi\,\mathrm{sgn}$	3	-1	-1	0	1

Theorem 4.20 (Taunt)**.** *If G has abelian p-Sylow groups, then $p \nmid |G' \cap \mathrm{Z}(G)|$.*

Proof. Assume indirectly that $U \le G' \cap \mathrm{Z}(G)$ with $|U| = p$ exists. Let $U \le P \in \mathrm{Syl}_p(G)$ and $1_U \ne \lambda \in \operatorname{Irr}(U)$. Let $\mu \in \operatorname{Irr}(P)$ with $(\mu, \lambda^P)_P \ne 0$. Then $(\mu_U, \lambda)_U \ne 0$ and $\mu_U = \lambda$ because $\mu(1) = 1$ (P abelian). We write

$$\mu^G = \sum_{\chi \in \operatorname{Irr}(G)} a_\chi \chi.$$

Because $p \nmid |G:P| = \mu^G(1)$, there exists a $\chi \in \operatorname{Irr}(G)$ with $a_\chi > 0$ and $p \nmid \chi(1) =: n$. In particular, $(\mu, \chi_P)_P \neq 0$ (Frobenius reciprocity). Thus $\mu_U = \lambda$ is also an irreducible constituent of χ_U. Let Δ be a representation with character χ. For $x \in U \subseteq \mathrm{Z}(G) \subseteq \mathrm{Z}(\chi)$, we then have $\Delta(x) = \lambda(x) 1_n$ (Theorem 2.14). It follows that $\chi_U = n\lambda$. Since $G/\operatorname{Ker}(\det \Delta) \leq \mathbb{C}^\times$ is abelian, $G' \subseteq \operatorname{Ker}(\det \Delta)$ holds. Because $U \subseteq G'$, it follows in particular that

$$1 = \det \Delta(x) = \lambda(x)^n$$

for all $x \in U$. On the other hand, $\lambda(x)^p = \lambda(x^p) = 1$ also holds. Because $\gcd(p, n) = 1$, there exist $\alpha, \beta \in \mathbb{Z}$ with $\alpha p + \beta n = 1$. One obtains: $\lambda(x) = (\lambda(x)^p)^\alpha (\lambda(x)^n)^\beta = 1$ for all $x \in U$. This contradicts $\lambda \neq 1_U$. □

5 Frobenius Groups

Definition 5.1. A *group action* of G on a non-empty set Ω is a homomorphism $f : G \to \operatorname{Sym}(\Omega)$. We write ${}^g\omega := (f(g))(\omega)$ for $g \in G$ and $\omega \in \Omega$. One also says: G *acts* on Ω.

Theorem 5.2 (Brauer's Permutation Lemma)**.** *Let H be a finite group such that G acts on $\operatorname{Cl}(H)$ and $\operatorname{Irr}(H)$. For all $g \in G$, $C \in \operatorname{Cl}(H)$ and $\chi \in \operatorname{Irr}(H)$, let ${}^g\chi({}^gC) = \chi(C)$ hold. Then the cycle type of $g \in G$ in $\operatorname{Sym}(\operatorname{Cl}(H))$ coincides with the cycle type of g in $\operatorname{Sym}(\operatorname{Irr}(H))$. In particular,*

$$|\{C \in \operatorname{Cl}(H) : {}^gC = C\}| = |\{\chi \in \operatorname{Irr}(H) : {}^g\chi = \chi\}|$$

for all $g \in G$.

Proof. Let $\operatorname{Cl}(H) = \{C_1, \ldots, C_k\}$ and $\operatorname{Irr}(H) = \{\chi_1, \ldots, \chi_k\}$. Let $X := (\chi_i(C_j))_{i,j}$ be the character table of H. Let $g \in G$ be fixed. The action of g on $\operatorname{Cl}(H)$ (resp. $\operatorname{Irr}(H)$) is then described by a permutation matrix P (resp. Q). Here $QX = ({}^g\chi_i(C_j)) = (\chi_i({}^{g^{-1}}C_j)) = XP$ holds. Since X is invertible (Remark 2.1), $Q = XPX^{-1}$ holds, i. e. Q and P are similar. Let $(l_1, \ldots, l_n)$ be the cycle type of P. According to Exercise 13, the eigenvalues of P are given by: $\{e^{2\pi i j/l_1} : j = 0, \ldots, l_1 - 1\} \cup \ldots \cup \{e^{2\pi i j/l_n} : j = 0, \ldots, l_n - 1\}$ (with multiplicities). Since P and Q have the same eigenvalues, $(l_1, \ldots, l_n)$ is also the cycle type of Q. The last assertion is obtained by counting cycles of length one. □

Definition 5.3. A finite group G is called a *Frobenius group*, if a subgroup $1 < H < G$ with $H \cap gHg^{-1} = 1$ for all $g \in G \setminus H$ exists. One calls H a *Frobenius complement*.

Example 5.4.

(i) Let $P \in \operatorname{Syl}_p(G)$ with $|P| = p$ and $\mathrm{N}_G(P) = P < G$. Then G is obviously a Frobenius group with Frobenius complement P. In particular, S_3 is a Frobenius group.

(ii) Let K be a finite field with $|K| > 2$. For $a \in K^\times$ and $b \in K$ we define $f_{a,b} : K \to K$, $x \mapsto ax + b$. Then

$$\operatorname{Aff}(1, K) := \{f_{a,b} : a \in K^\times, b \in K\} \leq \operatorname{Sym}(K)$$

is a Frobenius group (Exercise 15).

Remark 5.5. In the following, we want to show that a Frobenius complement H of G always possesses a normal complement N, i. e. $G = HN$ and $H \cap N = 1$.

Lemma 5.6. *Let H be a Frobenius complement in G. We set*

$$N := G \setminus \bigcup_{g\in G} gHg^{-1} \cup \{1\}.$$

Then $|N| = |G : H|$ (as a set). If $M \trianglelefteq G$ with $H \cap M = 1$, then $M \subseteq N$ follows.

Proof. For $x, y \in G$ we have

$$xHx^{-1} = yHy^{-1} \iff y^{-1}x \in \mathrm{N}_G(H) = H \iff xH = yH.$$

In the case $xHx^{-1} \neq yHy^{-1}$ we have $|xHx^{-1} \cap yHy^{-1}| = |x(H \cap x^{-1}yHy^{-1}x)x^{-1}| = |H \cap x^{-1}yHy^{-1}x| = 1$. This shows

$$\Big|\bigcup_{g\in G} gHg^{-1}\Big| = |G : H|(|H| - 1) + 1.$$

It follows that $|N| = |G| - |G : H|(|H| - 1) = |G : H|$. For $M \trianglelefteq G$ with $H \cap M = 1$ we also have $gHg^{-1} \cap M = g(H \cap M)g^{-1} = 1$ for all $g \in G$. Thus $M \subseteq N$. □

Lemma 5.7. *Let H be a Frobenius complement in G, and let $\psi \in \mathrm{CF}(H)$ with $\psi(1) = 0$. Then $(\psi^G)_H = \psi$.*

Proof. Let $1 \neq h \in H$ and $g \in G$ with $ghg^{-1} \in H$. Then $1 \neq ghg^{-1} \in H \cap gHg^{-1}$ and $g \in H$. By definition, we thus have

$$\psi^G(h) = \frac{1}{|H|}\sum_{g\in H} \psi(ghg^{-1}) = \frac{1}{|H|}\sum_{g\in H} \psi(h) = \psi(h).$$

Furthermore, $\psi^G(1) = |G : H|\psi(1) = 0 = \psi(1)$. □

Theorem 5.8 (Frobenius)**.** *Let G be a Frobenius group with Frobenius complement H. Then there exists an $N \trianglelefteq G$ with $G = HN$ and $H \cap N = 1$.*

Proof. Let $1_H \neq \psi \in \mathrm{Irr}(H)$ and $\theta := \psi - \psi(1)1_H$. Then certainly $\theta \in \mathrm{CF}(H)$ and $\theta(1) = 0$. According to Lemma 5.7,

$$1 + \psi(1)^2 = (\theta, \theta)_H = (\theta, (\theta^G)_H)_H = (\theta^G, \theta^G)_G.$$

Furthermore, $(\theta^G, 1_G)_G = (\theta, 1_H)_H = -\psi(1)$. Consequently, $\widetilde{\psi} := \theta^G + \psi(1)1_G \in \mathrm{CF}(G)$ with

$$(\widetilde{\psi}, \widetilde{\psi})_G = (\theta^G, \theta^G)_G + 2\psi(1)(\theta^G, 1_G)_G + \psi(1)^2 = 1.$$

Like θ, θ^G and $\widetilde{\psi}$ are also virtual characters (Remark 1.26). Thus $\pm\widetilde{\psi} \in \mathrm{Irr}(G)$. For $h \in H$, it holds that

$$\widetilde{\psi}(h) = \theta^G(h) + \psi(1) = \theta(h) + \psi(1) = \psi(h).$$

In particular, $\widetilde{\psi}(1) = \psi(1) > 0$. This shows $\widetilde{\psi} \in \mathrm{Irr}(G)$ with $\widetilde{\psi}_H = \psi$. We additionally set $\widetilde{1_H} := 1_G$. Let

$$M := \bigcap_{\psi\in\mathrm{Irr}(H)} \mathrm{Ker}(\widetilde{\psi}) \trianglelefteq G.$$

Then

$$M \cap H \subseteq \bigcap_{\psi\in\mathrm{Irr}(H)} \mathrm{Ker}(\psi) = 1$$

according to Exercise 9. According to Lemma 5.6, it follows that $M \subseteq N$. Conversely, for $g \in N$, it holds that

$$\widetilde{\psi}(g) - \widetilde{\psi}(1) = \widetilde{\psi}(g) - \psi(1) = \theta^G(g) = \frac{1}{|H|} \sum_{\substack{x \in G,\\ g \in x^{-1}Hx}} \theta(xgx^{-1}) = 0$$

for all $\psi \in \operatorname{Irr}(H)$. This shows $N \subseteq M$ and $N = M \trianglelefteq G$. Because $|N| = |G : H|$, it also holds that $|HN| = |H||N| = |G|$ and $G = HN$. $\square$

Definition 5.9. In the situation of Theorem 5.8, N is called the *Frobenius kernel* of G.

Remark 5.10.

(i) No proof of Theorem 5.8 is known that manages without character theory.

(ii) Thompson has shown that the Frobenius kernel N is always nilpotent, i. e., N is the direct product of its Sylow subgroups.

Theorem 5.11. *Let G be a Frobenius group with Frobenius kernel N. Then*

$$\operatorname{Irr}(G) = \operatorname{Irr}(G/N) \cup \{\psi^G : 1_N \neq \psi \in \operatorname{Irr}(N)\}.$$

Proof. Let H be a Frobenius complement of G. Then H acts on $\operatorname{Cl}(N)$ by ${}^hC := \{hxh^{-1} : x \in C\}$. One also easily shows that H acts on $\operatorname{Irr}(N)$ by conjugation. In this case, ${}^h\psi({}^hC) = \psi(C)$ holds for $h \in H$, $\psi \in \operatorname{Irr}(N)$ and $C \in \operatorname{Cl}(N)$. We count the fixed points of $h \in H \setminus \{1\}$ on $\operatorname{Cl}(N)$. First, let $hxh^{-1} = x \in N$. Then $x^{-1}hx = h \in H \cap x^{-1}Hx$ and $x = 1$. The orbits of $\langle h \rangle$ on $N \setminus \{1\}$ therefore have length $|\langle h \rangle|$. If $C \in \operatorname{Cl}(N) \setminus \{\{1\}\}$ is a fixed point of $\langle h \rangle$, then $|\langle h \rangle| \mid |C|$ follows. On the other hand, $|C| \mid |N|$. This contradicts Exercise 16. Thus $\{1\}$ is the only fixed point of $\langle h \rangle$ on $\operatorname{Cl}(N)$. By Brauer's permutation lemma, it follows that

$$\{\psi \in \operatorname{Irr}(N) : {}^h\psi = \psi\} = \{1_N\}$$

for all $h \neq 1$. Let $1_N \neq \psi \in \operatorname{Irr}(N)$. Then $G_\psi = N$ and $\psi^G \in \operatorname{Irr}(G)$ by Theorem 4.13. By Theorem 4.11, there exists an $e \in \mathbb{N}$ such that $(\psi^G)_N = e \sum_{gN \in G/N} {}^g\psi$. If $\psi, \psi_1 \in \operatorname{Irr}(N)$ are not conjugate, then $\psi^G \neq \psi_1^G$. For a $\chi \in \operatorname{Irr}(G/N)$, it is obvious that $\chi_N = \chi(1)1_N$ and thus $\chi \notin \{\psi^G : 1_N \neq \psi \in \operatorname{Irr}(N)\}$. Let $\mathcal{R}$ be a transversal for the conjugacy classes of $\operatorname{Irr}(N) \setminus \{1_N\}$ under H. Then

$$\begin{aligned}\sum_{\chi \in \operatorname{Irr}(G/N)} \chi(1)^2 + \sum_{\psi \in \mathcal{R}} \psi^G(1)^2 &= |G/N| + |G/N|^2 \sum_{\psi \in \mathcal{R}} \psi(1)^2 = |G/N| + |G/N| \sum_{h \in H} \sum_{\psi \in \mathcal{R}} ({}^h\psi)(1)^2 \\ &= |G/N| + |G/N| \sum_{1_N \neq \psi \in \operatorname{Irr}(N)} \psi(1)^2 = |G/N| + |G/N|(|N| - 1) = |G|.\end{aligned}$$

Thus we have found all irreducible characters of G. $\square$

6 Induction Theorems

Remark 6.1.

(i) Often one constructs the character table of G by inducing characters from $H \leq G$. We will see in this chapter which subgroups H must be considered for this purpose.

(ii) For subgroups $H, K \leq G$, $H \times K$ acts on G by ${}^{(h,k)}g = hgk^{-1}$ for $h \in H$, $k \in K$ and $g \in G$. The orbits HgK are called *double cosets*. We denote the set of double cosets by $H\backslash G/K$.

Theorem 6.2 (Mackey Formula)**.** *Let $H, K \leq G$ and $\varphi \in \mathrm{CF}(H)$. Then*

$$(\varphi^G)_K = \sum_{KgH \in K\backslash G/H} (({}^g\varphi)_{K\cap gHg^{-1}})^K.$$

Proof. Let R be a system of representatives for $K\backslash G/H$ and for $r \in R$ let S_r be a system of representatives for $K/K \cap rHr^{-1}$. For every $k \in K$ there exist $s \in S_r$ and $x \in K \cap rHr^{-1}$ with $k = sx$ and $krH = sxrH = sr(r^{-1}xr)H = srH$. For $s, t \in S_r$ it holds that

$$srK = trK \iff s(K \cap rHr^{-1}) = K \cap srHr^{-1} = K \cap trHr^{-1} = t(K \cap rHr^{-1}) \iff s = t.$$

This shows

$$G = \dot{\bigcup_{r\in R}} KrH = \dot{\bigcup_{r\in R}}\dot{\bigcup_{s\in S_r}} srH \qquad \text{(disjoint)}.$$

According to Remark 4.4, for $x \in K$ it holds that:

$$\begin{aligned}(\varphi^G)(x) &= \sum_{\substack{gH\in G/H,\\ xgH=gH}} {}^g\varphi(x) = \sum_{r\in R}\sum_{\substack{s\in S_r,\\ xsrH=srH}} {}^{sr}\varphi(x)\\ &= \sum_{r\in R}\sum_{\substack{s\in S_r,\\ xs(rHr^{-1})=s(rHr^{-1})}} {}^s({}^r\varphi)(x) = \sum_{r\in R}(({}^r\varphi)_{K\cap gHg^{-1}})^K(x).\end{aligned}$$

□

Definition 6.3.

(i) A group H is called *(p-)quasielementary* for a prime p if H has a cyclic normal subgroup N with $p \nmid |N|$, such that H/N is a p-group (i. e. N is a p'-Hall normal subgroup).

(ii) A group H is called *(p-)elementary* for a prime p if H is a direct product of a p-Sylow group and a cyclic group. Obviously, elementary groups are also quasielementary.

Lemma 6.4. *For every prime p and every $x \in G$ there exists a p-quasielementary subgroup $H \leq G$ with $p \nmid 1_H^G(x) \in \mathbb{Z}$.*

Proof. We write $\langle x\rangle = P \times Q$ with $P \in \mathrm{Syl}_p(\langle x\rangle)$ and choose $H/Q \in \mathrm{Syl}_p(\mathrm{N}_G(Q)/Q)$. Since Q is cyclic, H is p-quasielementary. For $g \in G$ with $gxg^{-1} \in H$, it follows that $gQg^{-1} \subseteq H$. Thus $gQg^{-1} \subseteq \{y \in H : p \nmid |\langle y\rangle\} = Q$ and $g \in \mathrm{N}_G(Q)$. This shows

$$\begin{aligned}1_H^G(x) &= \frac{1}{|H|}\sum_{\substack{g\in G,\\ gxg^{-1}\in H}} 1 = \frac{1}{|H|}|\{g \in \mathrm{N}_G(Q) : gxg^{-1} \in H\}| = \frac{1}{|H|}|\{g \in \mathrm{N}_G(Q) : g^{-1}xg \in H\}|\\ &= |\{gH \in \mathrm{N}_G(Q)/H : xgH = gH\}|.\end{aligned}$$

We must therefore count the fixed points of the action $\alpha : \langle x\rangle \to \mathrm{Sym}(\mathrm{N}_G(Q)/H)$ by left multiplication. Because of $QgH = gQH = gH$ for $g \in \mathrm{N}_G(Q)$, we have $Q \leq \mathrm{Ker}(\alpha)$. Since $xQ \in \langle x\rangle/Q \cong P$ is a p-element, $\alpha(x)$ is also a p-element. In particular, $\alpha(x)$ decomposes into cycles whose lengths are powers of p. This yields

$$1_H^G(x) = |\{gH \in \mathrm{N}_G(Q)/H : xgH = gH\}| \equiv |\mathrm{N}_G(Q)/H| \not\equiv 0 \pmod p.$$

□

Theorem 6.5 (Solomon)**.** *There exist* $a_H \in \mathbb{Z}$ *with*

$$1_G = \sum_{\substack{H \le G,\\ H \text{ quasielementary}}} a_H 1_H^G.$$

Proof. Let

$$\mathcal{Q}(G) := \Big\{ \sum_{H \le G \text{ quasiel.}} a_H 1_H^G : a_H \in \mathbb{Z} \Big\}.$$

For $x \in G$ let $U_x := \{\lambda(x) : \lambda \in \mathcal{Q}(G)\}$. Because $-\lambda \in \mathcal{Q}(G)$ for $\lambda \in \mathcal{Q}(G)$, U_x is a subgroup of $(\mathbb{Z}, +)$, i. e. $U_x = n\mathbb{Z}$ for some $n \in \mathbb{Z}$. According to Lemma 6.4, $U_x \nsubseteq p\mathbb{Z}$ for every prime p. This shows $U_x = \mathbb{Z}$. We choose $\lambda_x \in \mathcal{Q}(G)$ with $\lambda_x(x) = 1$ for $x \in G$. For quasielementary subgroups $H, K \le G$ and $g \in G$, $H \cap gKg^{-1} \le H$ is also quasielementary (Exercise 17). The Mackey formula thus shows

$$\begin{aligned} 1_H^G 1_K^G &\overset{4.5\text{(ii)}}{=} (1_H (1_K^G)_H)^G = ((1_K^G)_H)^G = \Big(\sum_{HgK \in H\backslash G/K} ((1_{gKg^{-1}})_{H \cap gKg^{-1}})^H \Big)^G \\ &\overset{4.5\text{(i)}}{=} \sum_{HgK \in H\backslash G/K} (1_{H \cap gKg^{-1}})^G \in \mathcal{Q}(G). \end{aligned}$$

In particular, $\mathcal{Q}(G)$ is closed under addition and multiplication. By expanding the equation

$$\prod_{x \in G} (1_G - \lambda_x) = 0,$$

one obtains a representation for 1_G in the desired form. □

Lemma 6.6. *Let* H *be elementary and* $\chi \in \mathrm{Irr}(H)$*. Then there exist* $K \le H$ *and* $\lambda \in \mathrm{Irr}(K)$ *with* $\lambda(1) = 1$ *and* $\lambda^H = \chi$*.*

Proof (Mann). Let H be a minimal counterexample. We write $H = P \times Q$ with $P \in \mathrm{Syl}_p(H)$. According to Theorem 2.5, $\chi = \psi\lambda$ with $\psi \in \mathrm{Irr}(P)$ and $\lambda \in \mathrm{Irr}(Q)$. In the case $Q \ne 1$, there exist $P_1 \le P$, $\psi_1 \in \mathrm{Irr}(P_1)$ with $\psi_1(1) = 1$ and $\psi = \psi_1^P$. Then $\chi = \psi_1^P(1_P\lambda) = (\psi_1\lambda)^H$ with $\psi_1\lambda \in \mathrm{Irr}(P_1 \times Q)$ according to Theorem 4.5(ii). Since Q is abelian, it also holds that $(\psi_1\lambda)(1) = \psi_1(1)\lambda(1) = 1$. This contradiction shows $Q = 1$, i. e. H is a p-group.

For $\lambda \in \mathrm{Irr}(H)$ with $\lambda(1) = 1$, we have $\chi\lambda \in \mathrm{Irr}(H)$ (Exercise 6). In the case $\chi\lambda = \chi$, it follows that $1 = (\chi, \chi\lambda)_H = (\chi\overline{\chi}, \lambda)_H$, i. e. λ is an irreducible constituent of $\chi\overline{\chi}$ with multiplicity 1. Write $\chi\overline{\chi} = 1_H + \lambda_1 + \ldots + \lambda_r + \sum_{i=1}^s a_i\psi_i$ with $\lambda_1(1) = \ldots = \lambda_r(1) = 1 < \psi_1(1) \le \ldots \le \psi_s(1)$. Then

$$r + 1 \equiv r + 1 + \sum_{i=1}^s a_i\psi_i(1) = (\chi\overline{\chi})(1) = \chi(1)^2 \equiv 0 \pmod p$$

and $r \ne 0$. Thus, let $1_H \ne \lambda \in \mathrm{Irr}(H)$ with $\chi\lambda = \chi$. For $x \in H \backslash \mathrm{Ker}(\lambda)$, we then have $(1_H - \lambda)(x)\chi(x) = (\chi - \lambda\chi)(x) = 0$ and thus $\chi(x) = 0$. Choose a maximal subgroup $M < H$ with $\mathrm{Ker}(\lambda) \subseteq M$. Because of $H' \subseteq \mathrm{Ker}(\lambda)$, it follows that $M \trianglelefteq H$. Since H/M contains an element of order p, $|H : M| = p$ follows from the maximality of M. Let $\psi \in \mathrm{Irr}(M)$ with $(\chi_M, \psi)_M \ne 0$. In the case $H_\psi = M$, we have $\psi^H = \chi$ according to Clifford. Because of $M < H$, there exist $M_1 \le M$ and $\psi_1 \in \mathrm{Irr}(M_1)$ with $\psi_1(1) = 1$ and $\psi_1^M = \psi$. According to Theorem 4.5(i), it then follows that $\psi_1^H = (\psi_1^M)^H = \psi^H = \chi$. This contradiction shows $H_\psi = H$. Thus $\chi_M = e\psi$ for some $e \in \mathbb{N}$ according to Theorem 4.11. Because of $\chi(1) = e\psi(1)$, e is a p-power. From Theorem 4.15 it now follows that $e = 1$, i. e. $\chi_M = \psi$. According to Gallagher,

$\psi^H = \chi_1 + \ldots + \chi_p$ with pairwise distinct $\chi_i \in \mathrm{Irr}(H)$ and $\chi_1 = \chi$. Furthermore, all χ_i have the same degree. In particular, $(\chi_i)_M = \psi$ for $i = 1, \ldots, p$. Since ψ^H and χ vanish on $H \setminus M$ ($\subseteq H \setminus \mathrm{Ker}(\lambda)$), $\chi_2 + \ldots + \chi_p$ also vanishes on $H \setminus M$. This yields the contradiction

$$0 = |H|(\chi_1, \chi_2 + \ldots + \chi_p)_H = \sum_{x \in M} \chi_1(x)\overline{(\chi_2 + \ldots + \chi_p)(x)} = |M|(p-1)(\psi, \psi)_M = |M|(p-1). \qquad \square$$

Theorem 6.7 (Brauer's Induction Theorem)**.** *For every (virtual) character χ of G, there exist $a_{H,\psi} \in \mathbb{Z}$ with*

$$\chi = \sum_{\substack{H \le G,\\ H \text{ elementar}}} \sum_{\substack{\psi \in \mathrm{Irr}(H),\\ \psi(1)=1}} a_{H,\psi}\psi^G.$$

Proof. Obviously we can assume that χ is irreducible. According to Lemma 6.6 we can neglect the condition $\psi(1) = 1$. If one has found the desired representation for the character 1_G, then one obtains a corresponding representation for χ by multiplication with χ (note: $\chi\psi^G = (\chi_H\psi)^G$). We can therefore assume $\chi = 1_G$. According to Solomon we can also assume that G is p-quasielementary for a prime p. Let G be a minimal counterexample, and let $P \in \mathrm{Syl}_p(G)$. Then P possesses a cyclic, normal complement N in G.

We consider the elementary subgroup $H := P\,\mathrm{C}_N(P) \cong P \times \mathrm{C}_N(P)$. Since G is not elementary, $H < G$ holds. Because of $(1_H^G, 1_G)_G = (1_H, 1_H)_H = 1$, $\zeta := 1_H^G - 1_G$ is a character of G. If every irreducible constituent of ζ is induced from a proper subgroup, then one easily obtains a desired representation for $1_G = 1_H^G - \zeta$ from the minimality of G. Consequently, there exists a $\psi \in \mathrm{Irr}(G)$ with $(\zeta, \psi)_G \neq 0$, such that ψ is not induced from a proper subgroup. For $g \in G$, $NgH = gNH = G$ holds. According to Mackey, we therefore have $1_N + \zeta_N = (1_H^G)_N = 1_{N\cap H}^N$. This shows $(1_N + \zeta_N, 1_N)_N = (1_{N\cap H}^N, 1_N)_N = (1_{N\cap H}, 1_{N\cap H})_{N\cap H} = 1$ and $(\psi_N, 1_N)_N \le (\zeta_N, 1_N)_N = 0$. We can therefore choose $1_N \neq \lambda \in \mathrm{Irr}(N)$ with $(\psi_N, \lambda)_N \neq 0$. Write $N = \langle x \rangle$. For $g \in G$, then $gxg^{-1} = x^r$ for some $r \in \mathbb{Z}$ because of $N \trianglelefteq G$. Furthermore, there exists an $s \in \mathbb{Z}$ with $N \cap H = \mathrm{C}_N(P) = \langle x^s \rangle$. Then $gx^sg^{-1} = x^{sr} = (x^s)^r$. This shows $\mathrm{C}_N(P) \trianglelefteq G$. For $a \in \mathrm{C}_N(P) \subseteq H$, we therefore have

$$1_H^G(a) = \frac{1}{|H|} \sum_{\substack{g \in G,\\ gag^{-1} \in H}} 1 = |G : H| = 1_H^G(1).$$

Thus $\mathrm{C}_N(P) \subseteq \mathrm{Ker}(1_H^G) = \mathrm{Ker}(\zeta) \subseteq \mathrm{Ker}(\psi)$ according to Exercise 9. Analogously, $\mathrm{C}_N(P) \subseteq \mathrm{Ker}(\psi) \cap N = \mathrm{Ker}(\psi_N) \subseteq \mathrm{Ker}(\lambda)$. Since ψ is not induced from any proper subgroup, $G_\lambda = G$ follows according to Clifford. In particular, $\lambda(y^{-1}xy) = {}^y\lambda(x) = \lambda(x)$ for $y \in P$. Thus $yxy^{-1} \in x\,\mathrm{Ker}(\lambda)$ (note: $\lambda(1) = 1$). As above, one shows $\mathrm{Ker}(\lambda) \trianglelefteq G$ (as a subgroup of the cyclic normal subgroup N). Consequently, P acts on the coset $x\,\mathrm{Ker}(\lambda)$ by conjugation. Counting the orbits yields

$$|x\,\mathrm{Ker}(\lambda) \cap \mathrm{C}_N(P)| \equiv |x\,\mathrm{Ker}(\lambda)| = |\mathrm{Ker}(\lambda)| \not\equiv 0 \pmod p.$$

This shows $\varnothing \neq x\,\mathrm{Ker}(\lambda) \cap \mathrm{C}_N(P) \subseteq x\,\mathrm{Ker}(\lambda) \cap \mathrm{Ker}(\lambda)$ and $x \in \mathrm{Ker}(\lambda)$. Thus $N = \langle x \rangle \subseteq \mathrm{Ker}(\lambda)$ and $\lambda = 1_N$. Contradiction. $\square$

Theorem 6.8. *A class function χ of G is an irreducible character if and only if the following conditions hold:*

(i) *For every elementary subgroup $H \le G$, χ_H is a virtual character.*

(ii) $(\chi, \chi)_G = 1$.

(iii) $\chi(1) > 0$.

Proof. For $\chi \in \mathrm{Irr}(G)$, (i)–(iii) obviously hold. Conversely, let $\chi \in \mathrm{CF}(G)$ such that (i)–(iii) hold. According to Theorem 6.7, there exist $a_{H,\psi} \in \mathbb{Z}$ with

$$1_G = \sum_{\substack{H \le G,\\ H \text{ elementar}}} \sum_{\psi \in \mathrm{Irr}(H)} a_{H,\psi} \psi^G. \tag{$*$}$$

By assumption, $\psi\chi_H$ and $\psi^G\chi = (\psi\chi_H)^G$ are virtual characters for $\psi \in \mathrm{Irr}(H)$. Multiplying $(*)$ by χ, one sees that χ is a virtual character of G. Thus there exist $a_\varphi \in \mathbb{Z}$ with $\chi = \sum_{\varphi \in \mathrm{Irr}(G)} a_\varphi \varphi$. Because of $1 = (\chi, \chi)_G = \sum_{\varphi \in \mathrm{Irr}(G)} a_\varphi^2$, it follows that $\pm\chi \in \mathrm{Irr}(G)$. From $\chi(1) > 0$, it finally follows that $\chi \in \mathrm{Irr}(G)$. $\square$

Theorem 6.9 (Artin). *For every character χ of G, there exist $a_{C,\psi} \in \mathbb{Q}$ with*

$$\chi = \sum_{\substack{C \le G,\\ C \text{ cyclic}}} \sum_{\psi \in \mathrm{Irr}(C)} a_{C,\psi} \psi^G.$$

Proof. As in Theorem 6.7, we can assume $\chi = 1_G$. We define an equivalence relation on G by

$$x \approx y :\Longleftrightarrow \exists g \in G : \langle x \rangle = g\langle y \rangle g^{-1}.$$

For an equivalence class K of $\approx$, it suffices to show that the characteristic function

$$\chi_K(x) := \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{if } x \notin K \end{cases}$$

has the desired form, because χ is the sum of all χ_K.

Let $x \in K$ be fixed. We argue by induction on $n := |\langle x \rangle|$. For $n = 1$, we have $K = \{1\}$ and $\chi_K = |G|^{-1} 1_1^G$ (see Example 4.8). So let $n > 1$. For $C := \langle x \rangle$ and $y \in G$, we have

$$1_C^G(y) = |\{gC \in G/C : g^{-1}yg \in C\}|$$

according to Remark 4.4. Let $z \in G$ with $\langle z \rangle = \langle y \rangle$. Then

$$g^{-1}yg \in C \Longleftrightarrow g^{-1}\langle y \rangle g \le C \Longleftrightarrow g^{-1}\langle z \rangle g \le C \Longleftrightarrow g^{-1}zg \in C$$

for $g \in G$ and therefore $1_C^G(z) = 1_C^G(y)$. Since 1_C^G is also a class function, 1_C^G is even constant on the $\approx$-classes. We choose $a \in \mathbb{Q}$ with $a1_C^G(x) = 1$. If no conjugate of y lies in C, then clearly $1_C^G(y) = 0$. Now let $\langle y \rangle < C$ and let K_y be the $\approx$-class of y. By induction, χ_{K_y} can be written in the desired form. Thus, $-a1_C^G\chi_{K_y}$ can also be written in the desired form. Summing over these functions, it follows that

$$\beta(y) := \begin{cases} -a1_C^G(y) & \text{if } \langle y \rangle < gCg^{-1} \text{ for some } g \in G, \\ 0 & \text{otherwise} \end{cases}$$

also has the desired form. Finally, $\chi_K = a1_C^G + \beta$ also has the desired form. $\square$

Remark 6.10.

(i) In the following, we deal with *K-representations*, i. e., homomorphisms $\Delta\colon G \to \mathrm{GL}(n, K)$, where K is an arbitrary subfield of $\mathbb{C}$.

(ii) One easily sees that Maschke's Theorem also holds for K-representations (in the proof, one only needs $\operatorname{char}(K) \nmid |G|$). Every K-representation can thus be written as a direct sum of irreducible K-representations. The same applies to the K-characters.

(iii) Similarly, the following form of Schur's Lemma holds: If Δ and Γ are non-similar irreducible K-representations and $A\Delta(g) = \Gamma(g)A$ for all $g \in G$, then $A = 0$ (the full version of Schur's Lemma requires that K is algebraically closed).

(iv) Thus, part (i) of Lemma 1.18 also remains true for K-representations. For distinct irreducible K-characters χ and ψ, we therefore have $(\chi, \psi)_G = 0$ (see proof of Theorem 1.19). Furthermore, $(\chi, \chi)_G > 0$, but not necessarily $(\chi, \chi)_G = 1$ (Exercise 19).

Definition 6.11. One calls $\exp(G) := \min\{n \in \mathbb{N} : g^n = 1 \ \forall g \in G\}$ the *exponent* of G.

Remark 6.12. Division with remainder yields numbers $a, r \in \mathbb{Z}$ with $|G| = a\exp(G) + r$ and $0 \le r < \exp(G)$. Then $g^r = g^{a\exp(G)+r} = g^{|G|} = 1$ for all $g \in G$ by Lagrange. This shows $r = 0$ and $\exp(G) \bigm| |G|$.

Theorem 6.13 (Brauer). *Let $\Delta\colon G \to \mathrm{GL}(n, \mathbb{C})$ be a matrix representation and let $\zeta := e^{2\pi i/\exp(G)}$. Then Δ can be realized over $\mathbb{Q}(\zeta)$, i. e. by a suitable choice of basis one can assume $\Delta\colon G \to \mathrm{GL}(n, \mathbb{Q}(\zeta))$.*

Proof. Let χ be the character of Δ. Let $K := \mathbb{Q}(\zeta)$. By Brauer's induction theorem, there exist elementary subgroups $H_1, \ldots, H_m$ and $\lambda_i \in \operatorname{Irr}(H_i)$ with $\lambda_i(1) = 1$ and

$$\chi = \sum_{i=1}^{m} a_i \lambda_i^G$$

for certain $a_i \in \mathbb{Z}$. Because of $\lambda_i(1) = 1$, we have $\lambda_i(h)^{\exp(G)} = \lambda_i(h^{\exp(G)}) = \lambda_i(1) = 1$ for $h \in H_i$. Therefore, λ_i can be realized over K. By Exercise 14, λ_i^G can therefore also be realized over K. We write

$$\lambda_i^G = \sum_{j=1}^{k} b_{ij} \tau_j,$$

where $\tau_1, \ldots, \tau_k$ are the irreducible K-characters of G. Then

$$\chi = \sum_{j=1}^{k} \sum_{i=1}^{m} a_i b_{ij} \tau_j.$$

By Remark 6.10, $(\tau_j, \tau_j)_G \sum_{i=1}^m a_i b_{ij} = (\chi, \tau_j)_G \ge 0$ holds for $j = 1, \ldots, k$, because τ_j is obviously also a character (over $\mathbb{C}$). Thus $\sum_{i=1}^m a_i b_{ij} \ge 0$, and χ is a K-character. There exists therefore a K-representation Γ with character χ. By Theorem 1.21, Δ and Γ are similar. This shows the claim. □

Remark 6.14.

(i) The characters of cyclic groups cannot be realized over any smaller field than in Theorem 6.13.

(ii) Theorem 6.13 makes it possible to realize representations on the computer, because every element in $\mathbb{Q}(\zeta)$ can be uniquely written in the form $a_0 + a_1\zeta + \ldots + a_k\zeta^k$ with $k := \varphi(\exp(G)) - 1$ and $a_1, \ldots, a_k \in \mathbb{Q}$.

Example 6.15. Let Δ be a matrix representation with character χ. According to Theorem 6.13, we can assume $\Delta\colon G \to \mathrm{GL}(n, \mathbb{Q}(\zeta))$ with $\zeta := e^{2\pi i/|G|}$. Let $\alpha \in \mathcal{G} := \mathrm{Gal}(\mathbb{Q}(\zeta)|\mathbb{Q})$. Obviously, α then induces an automorphism on $\mathrm{GL}(n, \mathbb{Q}(\zeta))$ by $\alpha((x_{ij})_{i,j=1}^n) = (\alpha(x_{ij}))_{i,j=1}^n$. Consequently, ${}^\alpha\Delta := \alpha \circ \Delta\colon G \to \mathrm{GL}(n, \mathbb{Q}(\zeta))$, $g \mapsto \alpha(\Delta(g))$ is also a matrix representation of G. Let $m \in \mathbb{Z}$ with $\alpha(\zeta) = \zeta^m$. For the character ${}^\alpha\chi$ of ${}^\alpha\Delta$, it then holds that ${}^\alpha\chi(g) = \operatorname{Trace}\alpha(\Delta(g)) = \alpha(\chi(g)) = \chi(g^m)$ for $g \in G$ (see proof of Lemma 2.12 and Exercise 5). Conversely, it is known from Algebra 1 that for every $m \in \mathbb{Z}$ with $\gcd(m, |G|) = 1$, there exists an $\alpha \in \mathcal{G}$ with $\alpha(\zeta) = \zeta^m$ ($\mathcal{G} \cong (\mathbb{Z}/|G|\mathbb{Z})^\times$). In this case, $g \mapsto \chi(g^m)$ is therefore always a character of G. For $m = -1$, one obtains ${}^\alpha\chi = \overline{\chi}$. For $\chi \in \mathrm{Irr}(G)$, ${}^\alpha\chi \in \mathrm{Irr}(G)$ also holds, because

$$({}^\alpha\chi, {}^\alpha\chi)_G = \frac{1}{|G|}\sum_{g\in G}\alpha(\chi(g))\alpha(\chi(g^{-1})) = \alpha((\chi,\chi)_G) = \alpha(1) = 1.$$

If $\chi(g) \notin \mathbb{Z}$ for some $g \in G$, then there always exists an $\alpha \in \mathcal{G}$ with ${}^\alpha\chi \neq \chi$, because $\mathbb{Q}(\zeta)|\mathbb{Q}$ is a Galois extension. One calls χ and ${}^\alpha\chi$ *algebraically conjugate*. In this way, one can frequently construct characters.

Remark 6.16. A group H is called an *M-group*, if every irreducible character of H is the induction of a character of degree 1 (i.e., every irreducible character is *monomial*). According to Lemma 6.6, elementary groups are M-groups. We will conversely show that every M-group is solvable.

Lemma 6.17. *Let ψ be a character of $H \leq G$. Then*

$$\mathrm{Ker}(\psi^G) = \bigcap_{g\in G} g\,\mathrm{Ker}(\psi)g^{-1}.$$

Proof. It holds that

$$x \in \mathrm{Ker}(\psi^G) \iff \psi^G(x) = \psi^G(1) = |G:H|\psi(1) \iff \sum_{\substack{g\in G,\\ gxg^{-1}\in H}} \psi(gxg^{-1}) = |G|\psi(1).$$

For $x \in \mathrm{Ker}(\psi^G)$ we thus have

$$|G|\psi(1) = \Bigg|\sum_{\substack{g\in G,\\ gxg^{-1}\in H}} \psi(gxg^{-1})\Bigg| \leq \sum_{\substack{g\in G,\\ gxg^{-1}\in H}} |\psi(gxg^{-1})| \leq |G|\psi(1)$$

by Lemma 2.12. The Cauchy-Schwarz inequality (cf. proof of Lemma 2.12) implies $gxg^{-1} \in H$ and $\psi(gxg^{-1}) = \psi(x)$ for all $g \in G$. Thus

$$|G|\psi(1) = \sum_{g\in G}\psi(gxg^{-1}) = |G|\psi(x)$$

and $\psi(gxg^{-1}) = \psi(1)$ for all $g \in G$. This shows

$$x \in \mathrm{Ker}(\psi^G) \iff \forall g \in G : gxg^{-1} \in \mathrm{Ker}(\psi) \iff x \in \bigcap_{g\in G} g\,\mathrm{Ker}(\psi)g^{-1}. \qquad \square$$

Definition 6.18. We define $G^{(1)} := G'$ and $G^{(i)} := (G^{(i-1)})'$ for $i \geq 2$.

Remark 6.19.

(i) As is well known, $G^{(1)} = G' \trianglelefteq G$. Let us assume inductively that $G^{(i-1)} \trianglelefteq G$ holds. Let $g \in G$. Then the map $G^{(i-1)} \to G^{(i-1)}$, $x \mapsto gxg^{-1}$ is an automorphism $\alpha \in \mathrm{Aut}(G^{(i-1)})$. By Remark 2.9 it follows that $gG^{(i)}g^{-1} = \alpha((G^{(i-1)})') = (G^{(i-1)})' = G^{(i)}$. Thus all $G^{(i)}$ are normal in G.

(ii) As is well known, G is solvable if and only if there exists a $k \in \mathbb{N}$ with $G^{(k)} = 1$.

Theorem 6.20 (Taketa). *Every finite M-group is solvable.*

Proof. Let G be an M-group. Let $1 = \alpha_1 < \alpha_2 < \ldots < \alpha_k$ be the degrees of the irreducible characters of G (without multiplicities). We first show $G^{(i)} \subseteq \mathrm{Ker}(\chi)$ for all $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = \alpha_i$. For $i = 1$, χ has degree 1 and $G^{(1)} = G' \subseteq \mathrm{Ker}(\chi)$ holds. Now let $i > 1$. We argue by induction on i. By assumption, there exist $H < G$ and $\lambda \in \mathrm{Irr}(H)$ with $\lambda(1) = 1$ and $\chi = \lambda^G$. Because $H < G$ and $(1_H^G, 1_G)_G = (1_H, 1_H)_H = 1$, 1_H^G is reducible. For every irreducible constituent ψ of 1_H^G, it thus holds that $\psi(1) < 1_H^G(1) = |G:H| = \lambda^G(1) = \chi(1)$. By induction, $G^{(i-1)} \subseteq \mathrm{Ker}(\psi)$. By Exercise 9 and Lemma 6.17, this also implies $G^{(i-1)} \subseteq \mathrm{Ker}(1_H^G) \subseteq H$. It follows that $G^{(i)} = (G^{(i-1)})' \subseteq H' \subseteq \mathrm{Ker}(\lambda)$. Since $G^{(i)} \trianglelefteq G$, Lemma 6.17 also shows

$$G^{(i)} \subseteq \bigcap_{g \in G} g\,\mathrm{Ker}(\lambda)g^{-1} = \mathrm{Ker}(\chi).$$

In particular,

$$G^{(k)} \subseteq \bigcap_{\chi \in \mathrm{Irr}(G)} \mathrm{Ker}(\chi) = 1$$

by Exercise 9, i. e. G is solvable. □

Remark 6.21. Not every solvable group is an M-group (see Exercise 20).

7 Frobenius-Schur Indicators

Remark 7.1. We concern ourselves with the question of how many "roots" an element $g \in G$ has.

Definition 7.2. For $n \in \mathbb{N}$ and $g \in G$ let

$$\theta_n(g) := |\{h \in G : h^n = g\}|.$$

Remark 7.3.

(i) In the case $\gcd(n, |G|) = 1$, it holds that $\theta_n(g) = 1$ for all $g \in G$, because the map $G \to G$, $g \mapsto g^n$ is a bijection.

(ii) Obviously θ_n is a class function, i. e. we can write

$$\theta_n = \sum_{\chi \in \mathrm{Irr}(G)} \nu_n(\chi)\chi$$

with $\nu_n(\chi) \in \mathbb{C}$.

Lemma 7.4. *For all $\chi \in \mathrm{Irr}(G)$ and $n \in \mathbb{N}$ we have*

$$\nu_n(\chi) = \frac{1}{|G|}\sum_{g\in G}\chi(g^n).$$

Proof. It holds that

$$\nu_n(\chi) = (\chi, \theta_n)_G = \frac{1}{|G|}\sum_{g\in G}\chi(g)\theta_n(g) = \frac{1}{|G|}\sum_{g\in G}\sum_{\substack{h\in G,\\ h^n=g}}\chi(h^n) = \frac{1}{|G|}\sum_{h\in G}\sum_{\substack{g\in G,\\ h^n=g}}\chi(h^n) = \frac{1}{|G|}\sum_{h\in G}\chi(h^n).\square$$

Definition 7.5. One calls $\nu_2(\chi)$ the *Frobenius-Schur indicator* of $\chi \in \mathrm{Irr}(G)$.

Theorem 7.6. *For $\chi \in \mathrm{Irr}(G)$ it holds:*

$$\nu_2(\chi) = \begin{cases} \pm 1 & \text{if } \overline{\chi} = \chi, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\Delta\colon G \to \mathrm{GL}(n, \mathbb{C})$, $g \mapsto (\gamma_{ij}(g))_{i,j=1}^n$ be a matrix representation with character χ. We consider the representation $\Delta^2 := \Delta \otimes \Delta\colon G \to \mathrm{GL}(n^2, \mathbb{C})$ (see Remark 2.2 and Remark 2.4). As there, let $i \mapsto (i_1, i_2)$ be a bijection between $\{1, \ldots, n^2\}$ and $\{1, \ldots, n\}^2$. It is easy to see that

$$U := \{(a_1, \ldots, a_{n^2}) \in \mathbb{C}^{n^2} : a_i = -a_j \text{ if } (i_1, i_2) = (j_2, j_1)\}$$

is a subspace of $\mathbb{C}^{n^2}$. For $(a_1, \ldots, a_{n^2}) \in U$ and $g \in G$, we have

$$(\Delta^2(g))(a_1, \ldots, a_{n^2}) = \left(\sum_{j=1}^{n^2} a_j \gamma_{i_1 j_1}(g)\gamma_{i_2 j_2}(g)\right)_{i=1}^{n^2}.$$

Let $k \in \{1, \ldots, n^2\}$ with $(k_1, k_2) = (i_2, i_1)$. Then

$$\sum_{j=1}^{n^2} a_j \gamma_{i_1 j_1}(g)\gamma_{i_2 j_2}(g) = -\sum_{j=1}^{n^2} a_j \gamma_{i_1 j_2}(g)\gamma_{i_2 j_1}(g) = -\sum_{j=1}^{n^2} a_j \gamma_{k_1 j_1}(g)\gamma_{k_2 j_2}(g).$$

Thus $(\Delta^2(g))(a_1, \ldots, a_{n^2}) \in U$ and U is Δ^2-invariant. By Maschke, there exists a Δ^2-invariant complement $V \leq \mathbb{C}^{n^2}$ of U. Let $\Lambda^2 : G \to \mathrm{GL}(U)$ and $S^2 : G \to \mathrm{GL}(V)$ be the corresponding subrepresentations (so $\Delta^2 = \Lambda^2 \oplus S^2$). wlog. let $i_1 < i_2$ for $i = 1, \ldots, n(n-1)/2$. We choose $i' \in \{1, \ldots, n^2\}$ with $(i'_1, i'_2) = (i_2, i_1)$. Then $\{b_i := e_i - e_{i'} : i = 1, \ldots, n(n-1)/2\}$ is a basis of U, where e_i is the standard basis of $\mathbb{C}^{n^2}$. Then

$$\begin{aligned}(\Lambda^2(g))(b_i) = (\Delta^2(g))(b_i) &= \big(\gamma_{j_1 i_1}(g)\gamma_{j_2 i_2}(g) - \gamma_{j_1 i_2}(g)\gamma_{j_2 i_1}(g)\big)_{j=1}^{n^2} \\ &= \sum_{j=1}^{\frac{n(n-1)}{2}} \big(\gamma_{j_1 i_1}(g)\gamma_{j_2 i_2}(g) - \gamma_{j_1 i_2}(g)\gamma_{j_2 i_1}(g)\big) b_j\end{aligned}$$

for $g \in G$ and $i = 1, \dots, n(n-1)/2$. For the character λ of Λ^2, it thus holds that

$$\lambda(g) = \sum_{i=1}^{\frac{n(n-1)}{2}} \gamma_{i_1 i_1}(g)\gamma_{i_2 i_2}(g) - \gamma_{i_1 i_2}(g)\gamma_{i_2 i_1}(g) = \frac{1}{2}\Big(\sum_{i=1}^{n} \gamma_{ii}(g)\Big)^2 - \frac{1}{2}\underbrace{\Big(\sum_{i,j=1}^{n} \gamma_{ij}(g)\gamma_{ji}(g)\Big)}_{\substack{=\mathrm{Trace}\,\Delta(g)^2\\ =\mathrm{Trace}\,\Delta(g^2)}}$$

$$= \frac{1}{2}\big(\chi(g)^2 - \chi(g^2)\big)$$

for $g \in G$. Since λ is a constituent of χ^2 ($\Delta^2 = \Lambda^2 \oplus S^2$), we have $0 \le (\lambda, 1_G)_G \le (\chi^2, 1_G)_G = (\chi, \overline{\chi})_G \le 1$. According to Lemma 7.4, it follows that

$$\nu_2(\chi) = (\chi^2 - 2\lambda, 1_G)_G = (\chi^2, 1_G)_G - 2(\lambda, 1_G)_G = (\chi, \overline{\chi})_G - 2(\lambda, 1_G)_G = \begin{cases} \pm 1 & \text{if } \overline{\chi} = \chi, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

Remark 7.7.

(i) The representation Λ^2 (resp. S^2) constructed in the proof is called the *alternating* (resp. *symmetric) square* of Δ.

(ii) It holds that $\nu_2(\chi) = 1$ if and only if an $\mathbb{R}$-representation with character χ exists (without proof).

Definition 7.8. An element of order 2 in G is called an *involution*.

Theorem 7.9. *The number of involutions in G is*

$$\sum_{\chi \in \mathrm{Irr}(G)} \nu_2(\chi)\chi(1) - 1 = \sum_{\substack{\chi \in \mathrm{Irr}(G),\\ \overline{\chi} = \chi \ne 1_G}} \nu_2(\chi)\chi(1).$$

Proof. This follows from $|\{x \in G : x^2 = 1\}| = \theta_2(1)$. $\square$

Lemma 7.10. *Let $t > 0$ be the number of involutions in G. Then there exists an $x \in G \setminus \{1\}$ with $|G : \mathrm{C}_G(x)| \le (|G|/t)^2$.*

Proof. Let $S := \{\chi \in \mathrm{Irr}(G) : 1_G \ne \chi = \overline{\chi}\}$. According to Theorem 7.9, $0 < t \le \sum_{\chi \in S} \chi(1)$. In particular, $S \ne \varnothing$. From the Cauchy-Schwarz inequality it follows that

$$t^2 \le \Big(\sum_{\chi \in S} \chi(1)\Big)^2 \le \sum_{\chi \in S} 1^2 \sum_{\chi \in S} \chi(1)^2 = |S| \sum_{\chi \in S} \chi(1)^2 \le |S||G| \le (k(G) - 1)|G|.$$

If every non-trivial conjugacy class of G had more than $(|G|/t)^2$ elements, then

$$|G| - 1 > (k(G) - 1)\frac{|G|^2}{t^2} \ge |G|. \qquad \square$$

Theorem 7.11 (Brauer-Fowler)**.** *Let $n \in \mathbb{N}$. Then there exist only finitely many simple groups G with an involution x such that $|\mathrm{C}_G(x)| \le n$ holds.*

Proof. If G is abelian, then $|G| = |\mathrm{C}_G(x)| \le n$ and the claim is clear. So let G be non-abelian. The conjugacy class of x in G contains $|G : \mathrm{C}_G(x)| \ge |G|/n$ elements. This is therefore a lower bound for the number of involutions in G. According to Lemma 7.10, there exists a $y \in G \setminus \{1\}$ with $|G : \mathrm{C}_G(y)| \le (|G|/(|G|/n))^2 = n^2$. Since G is simple and non-abelian, $\mathrm{C}_G(y) < G$ also holds. As usual, G acts transitively (and thus non-trivially) on $G/\mathrm{C}_G(y)$ by left multiplication (i. e. ${}^g(h\,\mathrm{C}_G(y)) := gh\,\mathrm{C}_G(y)$). The corresponding homomorphism $\alpha : G \to \mathrm{Sym}(G/\mathrm{C}_G(y))$ is then also non-trivial. Since G is simple, it follows that $\mathrm{Ker}(\alpha) = 1$. In particular, $|G| \le |\mathrm{Sym}(G/\mathrm{C}_G(y))| \le n^2!$. The claim follows. □

Remark 7.12. According to Feit-Thompson ("Groups of odd order are solvable"), every simple, non-abelian group possesses an involution. The Brauer-Fowler theorem was the fundamental idea behind the classification of finite simple groups.

Lemma 7.13. *Let p^n be a prime power. The sum of the primitive p^n-th roots of unity in $\mathbb{C}$ is 1 if $n = 0$, -1 if $n = 1$ or 0 if $n \ge 2$.*

Proof. We can assume $n \ge 1$. For $\zeta := e^{2\pi i/p^n} \in \mathbb{C}$ it then holds that $\sum_{i=0}^{p^n-1} \zeta^i = \frac{\zeta^{p^n}-1}{\zeta-1} = 0$. For $n = 1$ the sum of the primitive roots of unity is $\sum_{i=1}^{p-1} \zeta^i = -1$. For $n \ge 2$ it follows that

$$\sum_{\substack{0<i<p^n\\ \gcd(i,p)=1}} \zeta^i = \sum_{i=0}^{p^n-1} \zeta^i - \sum_{i=0}^{p^{n-1}-1} \zeta^{pi} = 0. \qquad \square$$

Theorem 7.14 (Frobenius)**.** *For all $n \in \mathbb{N}$ and $\chi \in \mathrm{Irr}(G)$ it holds that*

$$\frac{1}{\gcd(n, |G|)} \sum_{\substack{g\in G,\\ g^n=1}} \chi(g) \in \mathbb{Z}.$$

In particular, $\gcd(n, |G|) \mid \theta_n(1)$.

Proof. Let ρ be the regular character of G. We first show that $\tau(g) := \rho(g^n)/\gcd(n, |G|)$ for $g \in G$ is a virtual character of G. According to Brauer (see proof of Theorem 6.8), it suffices to show that τ_E is virtual for every elementary subgroup $E \le G$. Let ρ_1 be the regular character of E. For $g \in E$, we then have

$$\tau_E(g) = \frac{\rho(g^n)}{\gcd(n, |G|)} = \underbrace{\frac{|G : E| \gcd(n, |E|)}{\gcd(n, |G|)}}_{\in\mathbb{N}} \frac{\rho_1(g^n)}{\gcd(n, |E|)}.$$

Thus, we can assume $G = E$. Then G is the direct product of its Sylow subgroups $G = P_1 \times \ldots \times P_s$. Let ρ_i be the regular character of P_i. For $g = x_1 \ldots x_s \in G$ ($x_i \in P_i$), it then holds that

$$\tau(G) = \frac{\rho_1(x_1^n)}{\gcd(n, |P_1|)} \cdots \frac{\rho_s(x_s^n)}{\gcd(n, |P_s|)}.$$

By induction on s, we can assume that G is a p-group. Let $\gcd(n, |G|) = p^a$. In the case $p^{a+1} \mid n$, we have $|G| = p^a$ and $\tau(g) = \rho(1)/|G| = 1_G(g)$. We can therefore assume $n = p^a m$ with $p \nmid m$. Let $\tau'(g) := \rho(g^{p^a})/p^a$ for $g \in G$. If we can show that τ' is a virtual character of G, then this also holds for τ, because $\tau(g) = \tau'(g^m)$ for $g \in G$ (see Example 6.15). We can therefore assume $n = p^a$. For

$\chi \in \mathrm{Irr}(G)$, we must show: $(\chi, \tau)_G \in \mathbb{Z}$. According to Lemma 6.6, there exist $H \leq G$ and $\varphi \in \mathrm{Irr}(H)$ with $\varphi(1) = 1$ and $\chi = \varphi^G$. Thus, it suffices to show:

$$(\chi, \tau)_G = (\varphi, \tau_H)_H = \frac{|G:H|}{p^a} \sum_{\substack{h \in H, \\ h^{p^a}=1}} \varphi(h) \in \mathbb{Z}.$$

Let $x \in H$ with $|\langle x \rangle| = p^b > p^a$, and let $\varphi(x)$ be a primitive p^c-th root of unity. According to Lemma 7.13, we have

$$\sum_{\substack{h \in H, \\ \langle h \rangle = \langle x \rangle}} \varphi(h) = \begin{Bmatrix} p^{b-1}(p-1) & \text{if } c = 0, \\ -p^{b-1} & \text{if } c = 1, \\ 0 & \text{otherwise} \end{Bmatrix} \equiv 0 \pmod{p^a}.$$

Since $x \sim y :\Leftrightarrow \langle x \rangle = \langle y \rangle$ is an equivalence relation on H, the following congruence holds:

$$|G:H| \sum_{\substack{h \in H, \\ h^{p^a}=1}} \varphi(h) \equiv |G:H| \Bigg(\sum_{\substack{h \in H, \\ h^{p^a}=1}} \varphi(h) + \sum_{\substack{h \in H, \\ h^{p^a} \neq 1}} \varphi(h) \Bigg) = |G|(\varphi, 1_H)_H \equiv 0 \pmod{p^a}.$$

This finally shows that τ is a virtual character for every finite group G. Because $(\chi, \tau)_G \in \mathbb{Z}$ for $\chi \in \mathrm{Irr}(G)$, the first assertion follows. The second assertion is obtained by $\chi = 1_G$. $\square$

Remark 7.15.

(i) A similar proof shows $\gcd(n, |\mathrm{C}_G(g)|) \mid \theta_n(g)$ for $g \in G$.

(ii) Using the classification of finite simple groups, it was possible to prove the following conjecture of Frobenius:

$$\theta_n(1) = n \;\Big|\; |G| \Longrightarrow \{g \in G : g^n = 1\} \trianglelefteq G.$$

8 Normal Complements

Definition 8.1. Let $\mathbb{P}$ be the set of all prime numbers and $\pi \subseteq \mathbb{P}$. We set $\pi' := \mathbb{P} \setminus \pi$. An element $x \in G$ is called a *π-element*, if every prime divisor of $|\langle x \rangle|$ lies in π. Analogously, G is a *π-group*, if every prime divisor of $|G|$ lies in π.

Remark 8.2.

(i) According to Lagrange and Cauchy, G is a π-group if and only if every element in G is a π-element.

(ii) Let $x \in G$. As is well known, $\langle x \rangle$ possesses for every prime number p exactly one (normal) p-Sylow subgroup S_p. In particular, $\langle x \rangle = \prod_{p \in \mathbb{P}} S_p$. Consequently, there exist uniquely determined $x_p \in S_p$ with $x = \prod_{p \in \mathbb{P}} x_p$. One calls x_p the *p-factor* of x. For $\pi \subseteq \mathbb{P}$ let $x_\pi := \prod_{p \in \pi} x_p$. Then x_π is the *π-factor* of x. Obviously $x = x_\pi x_{\pi'}$.

Theorem 8.3 (Brauer-Dade)**.** *Let $N \trianglelefteq H \leq G$ and $\pi \subseteq \mathbb{P}$ with the following properties:*

(i) H/N is a π-group.

(ii) $|G:H|$ is not divisible by any prime number in π.

(iii) If $x, y \in H$ are conjugate in G, then xN, yN are conjugate in H/N.

(iv) If h is a π-element in $H \setminus N$ and $P \in \mathrm{Syl}_p(\mathrm{C}_G(h))$ for a prime number $p \in \pi$ with $p \nmid |\langle h \rangle|$, then $\langle h \rangle P$ is conjugate to a subgroup of H.

Then there exists a normal subgroup $M \trianglelefteq G$ with $G = HM$ and $H \cap M = N$.

Remark 8.4. If one has already proven the existence of M, then one can construct $\mathrm{Irr}(G/M)$ from $\mathrm{Irr}(H/N)$ and the isomorphism $G/M = HM/M \cong H/H \cap M = H/N$. In the proof of Theorem 8.3, one proceeds in the opposite direction and first constructs the inflation of the characters in $\mathrm{Irr}(G/M)$ and obtains M as the intersection of the kernels of these characters (cf. proof of Theorem 5.8).

Proof. Let $\mathrm{Cl}(H/N) = \{C_1, \ldots, C_k\}$ and $C_1 = \{1\}$. For $i = 1, \ldots, k$ let

$$B_i := \{h \in H : hN \in C_i\},$$

so $B_1 = N$ and $H = \bigcup_{i=1}^k B_i$. For $i = 2, \ldots, k$ let

$$A_i := \{g \in G : g_\pi \text{ is conjugate to an element in } B_i\}.$$

With $A_1 := G \setminus \bigcup_{i=2}^k A_i$, we then have $G = \bigcup_{i=1}^k A_i$.

For $i = 1, \ldots, k$, A_i is a union of conjugacy classes in G.

Claim 1: $G = \dot{\bigcup}_{i=1}^k A_i$.
Proof: Let $g \in A_i \cap A_j$ and wlog. $i \neq 1 \neq j$, then g_π is conjugate to an element $h_i \in B_i$ and an element $h_j \in B_j$. Consequently, there exists an $x \in G$ with $x h_i x^{-1} = h_j$. Because of (iii), $h_i N$ and $h_j N$ are then conjugate in H/N, i. e. $i = j$.

Claim 2: $A_i \cap H = B_i$.
Proof: First let $i \geq 2$ and $h = h_\pi h_{\pi'} \in B_i$. Then $h_{\pi'} \in N$ by (i), so $hN = h_\pi N$ and thus $h_\pi \in B_i$ and $h \in A_i \cap H$. Conversely, if $h \in A_i \cap H$, then h_π is conjugate to an element in B_i. Consequently, $hN = h_\pi N \in C_i$ because of (iii) and thus $h \in B_i$. By definition, $A_1 \cap H = H \setminus \bigcup_{i=2}^k (A_i \cap H) = H \setminus \bigcup_{i=2}^k B_i = B_1$.

For $p \in \pi$, H contains a p-Sylow subgroup of G because of (ii). Therefore, every p-element $x \in G$ is conjugate to an element $y \in H$. We say that x is of type I in the case $y \in H \setminus N$ and of type II in the case $y \in N$. By (iii), x is either of type I or of type II, but not both. For each $g \in G$ we set

$$\alpha(g) := \prod_{\substack{p \in \pi, \\ g_p \text{ of type I}}} g_p \qquad \text{and} \qquad \beta(g) := \prod_{\substack{p \in \pi, \\ g_p \text{ of type II}}} g_p.$$

Then $g = \alpha(g)\beta(g)g_{\pi'}$.

Claim 3: For every $g \in G$ with $\alpha(g) \neq 1$, g_π is conjugate to an element in $H \setminus N$.
Proof: If g_π is a p-element for a prime p, then g_π is of type I because of $\alpha(g) \neq 1$, and we are done. If g_π is not a p-element, then there exists a prime divisor p of $|\langle g_\pi \rangle|$ with $g_\pi = x g_p$ and $\alpha(x) \neq 1$ for $x := g_{\pi \setminus \{p\}}$. Consequently, there exists a $q \in \pi \setminus \{p\}$ such that g_q is of type I. Arguing by induction on the number of prime factors of $|\langle g_\pi \rangle|$, we can assume that x is conjugate to an element in $H \setminus N$. By replacing g with a conjugate, we can even assume $x \in H \setminus N$. Let $P \in \mathrm{Syl}_p(\mathrm{C}_G(x))$ with $g_p \in P$. By (iv), $\langle x \rangle P$ is conjugate to a subgroup of H. In particular, $g_\pi = x g_p$ is conjugate to an element $h \in H$. In the case $h \in N$, we would have $h_q \in N$. But then g_q would be of type II. Thus $h \notin N$, and the claim is proven.

Claim 4: $A_1 = \{g \in G : \alpha(g) = 1\}$.
Proof: Indeed, if $g \in G$ with $\alpha(g) \neq 1$, then $g \in A_i$ for some $i \in \{2, \ldots, k\}$ by Claim 3. Conversely, if $g \in A_i$ for some $i \in \{2, \ldots, k\}$, then g_π is conjugate to an element $h \in B_i$. Consequently, $\alpha(g)$ is conjugate to $\alpha(h)$ and $\beta(g)$ to $\beta(h)$. Every p-factor y of $\beta(h)$ thus lies in H and is conjugate to an element in N. By (iii), we have $y \in N$. This shows: $\beta(h) \in N$. Consequently, $\alpha(h)N = \alpha(h)\beta(h)N = hN \in C_i$ and thus $\alpha(h) \in B_i$. In particular, $\alpha(h) \neq 1$ and $\alpha(g) \neq 1$.

Claim 5: $g \in A_i \Longrightarrow \alpha(g), g_\pi \in A_i$.
Proof: For $g \in A_i$, $g_\pi \in A_i$ by the definition of A_i. For $i = 1$, $\alpha(g) = 1 \in B_1 \subseteq A_1$ by Claim 4. So let $i \geq 2$. Then, as above, $\alpha(g)$ is conjugate to an element $\alpha(h) \in B_i \subseteq A_i$, so also $\alpha(g) \in A_i$.

Claim 6: $\alpha(g) = 1 \Longrightarrow \alpha(g^n) = 1$ for $n \in \mathbb{N}$.
Proof: For $\alpha(g) = 1$, every p-factor (for $p \in \pi$) of g is conjugate to an element in N. Therefore, every p-factor of g^n is also conjugate to an element of N. This shows $\alpha(g^n) = 1$.

Claim 7: If $E = U \times Q \leq G$ is a π-subgroup with $U = \langle u \rangle$, $\alpha(u) \neq 1$ and $Q \in \mathrm{Syl}_p(E)$, then E is conjugate to a subgroup of H.
Proof: Because of $\alpha(u) \neq 1$, $u = u_\pi$ is conjugate to an element in $H \setminus N$ (Claim 3). By replacing E with a conjugate, we can thus assume $u \in H \setminus N$. Then the claim follows from (iv).

Let $\mathrm{Irr}(H/N) = \{\psi_1, \ldots, \psi_k\} \subseteq \mathrm{Irr}(H)$. For $i = 1, \ldots, k$, ψ_i is then constant on each B_j and possesses exactly one extension $\chi_i \in \mathrm{CF}(G)$ that is constant on each A_j. We show $\chi_i \in \mathrm{Irr}(G)$ using Theorem 6.8. For this, let $E = E_\pi \times E_{\pi'} \leq G$ be elementary, where E_π is a π-group and $E_{\pi'}$ is a π'-group. For $x \in E$, we have $x_\pi \in E_\pi$ and $\chi_i(x) = \chi_i(x_\pi)$, because x and x_π lie in the same A_j (Claim 5). Therefore, $(\chi_i)_E = (\chi_i)_{E_\pi} 1_{E_{\pi'}}$ and we can assume $E = E_\pi$ (Exercise 17). Let $E = U \times Q$ with $U = \langle u \rangle$ and $Q \in \mathrm{Syl}_p(E)$. In the case $\alpha(u) \neq 1$, we can replace E with a conjugate and assume $E \leq H$ (Claim 7). Then $(\chi_i)_E = (\psi_i)_E$ is a character of E. So let $\alpha(u) = 1$. Let $x = u^n v \in E$ with $v \in Q$. By Claim 6, $\alpha(x) = \alpha(u^m v) = \alpha(u^m)\alpha(v) = \alpha(v)$ and

$$\chi_i(x) = \chi_i(\alpha(x) \underbrace{\beta(x)}_{\in \mathrm{Ker}(\chi_i)}) = \chi_i(\alpha(x)) = \chi_i(\alpha(v)) = \chi_i(v).$$

This shows $(\chi_i)_E = 1_U (\chi_i)_Q$ and we can assume $E = Q$. By Sylow, we can assume $E \leq H$. But then $(\chi_i)_E = (\psi_i)_E$ is a character of E. Thus χ_i is a virtual character of G.

For $i = 1, \ldots, k$ let $b_i \in B_i$ and $\theta_i := \sum_{j=1}^k \psi_j(b_i^{-1})\chi_j$. According to the second orthogonality relation for H/N, we then have

$$\begin{aligned}(\theta_i, 1_G)_G &= \frac{1}{|G|}\sum_{g \in G} \theta_i(g) = \frac{1}{|G|}\sum_{g \in G}\sum_{j=1}^k \psi_j(b_i^{-1})\chi_j(g) = \frac{1}{|G|}\sum_{r=1}^k |A_r| \sum_{j=1}^k \psi_j(b_i^{-1})\psi_j(b_r) \\ &= \frac{|A_i|}{|G|}|\mathrm{C}_{H/N}(b_i N)| = \frac{|A_i|}{|G|}\frac{|H:N|}{|C_i|} = \frac{|A_i|}{|G:H| \cdot |C_i||N|} = \frac{|A_i|}{|G:H||B_i|} \in \mathbb{Q}.\end{aligned}$$

On the other hand, the multiplicities of the irreducible constituents of θ_i are clearly algebraic integers. By Lemma 3.5, we thus have $(\theta_i, 1_G)_G = \frac{|A_i|}{|G:H||B_i|} \in \mathbb{N}$ and $|A_i| \geq |G:H||B_i|$. Therefore,

$$|G| = \sum_{i=1}^k |A_i| \geq |G:H| \sum_{i=1}^k |B_i| = |G:H||H| = |G|.$$

For $i = 1, \dots, k$, we thus have $|A_i| = |G : H||B_i|$. Consequently,

$$(\chi_i, \chi_i)_G = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\chi_i(g^{-1}) = \frac{1}{|G|} \sum_{j=1}^{k} |A_j| \psi_i(b_j)\psi_i(b_j^{-1})$$
$$= \frac{1}{|H|} \sum_{j=1}^{k} |B_j| \psi_i(b_j)\psi_i(b_j^{-1}) = (\psi_i, \psi_i)_H = 1.$$

Because of $\chi_i(1) = \psi_i(1) > 0$, we thus have $\chi_i \in \mathrm{Irr}(G)$ by Theorem 6.8. Because of $B_1 = N = \bigcap_{i=1}^{k} \mathrm{Ker}(\psi_i)$, we have $A_1 \subseteq \bigcap_{i=1}^{k} \mathrm{Ker}(\chi_i)$. For $x \in B_i \subseteq A_i$ with $i \geq 2$, there exists a ψ_j with $\chi_j(x) = \psi_j(x) \neq \psi_j(1) = \chi_j(1)$. Thus $A_1 = \bigcap_{i=1}^{k} \mathrm{Ker}(\chi_i) \trianglelefteq G$ with $A_1 \cap H = B_1 = N$. Because of $|A_1| = |G : H||B_1| = |G : H||N|$, we have

$$|A_1 H| = \frac{|A_1||H|}{|A_1 \cap H|} = \frac{|G||N|}{|N|} = |G|.$$

This shows $A_1 H = G$, and we are done. □

Theorem 8.5 (Dade)**.** *Let $N \trianglelefteq H \leq G$ and $\pi \subseteq \mathbb{P}$ with the following properties:*

(i) H/N is a π-group.

(ii) If $x, y \in H$ are conjugate in G, then xN, yN are conjugate in H/N.

(iii) Every elementary π-subgroup of G is conjugate to a subgroup of H.

Then there exists a normal subgroup M of G with $G = HM$ and $H \cap M = N$.

Proof. Let $p \in \pi$ and $P \in \mathrm{Syl}_p(G)$. Since P is elementary, P is conjugate to a subgroup of H, so $p \nmid |G : H|$. Thus G satisfies the assumptions of Theorem 8.3, and we are finished. □

Theorem 8.6 (Brauer-Suzuki)**.** *Let $\pi \subseteq \mathbb{P}$ and H a π-subgroup of G with the following properties:*

(i) If $x, y \in H$ are conjugate in G, then they are also conjugate in H.

(ii) Every elementary π-subgroup of G is conjugate to a subgroup of H.

Then there exists a normal subgroup M of G with $G = HM$ and $H \cap M = 1$.

Proof. Theorem 8.5 with $N := 1$. □

Lemma 8.7. *Let P be a p-group and $U < P$. Then $U < \mathrm{N}_P(U)$.*

Proof. We argue by induction on $|P|$. In the case $|P| = p$, we have $1 = U < \mathrm{N}_P(U) = P$. So let $|P| > p$. Because of $\mathrm{Z}(P) \subseteq \mathrm{N}_P(U)$, we can assume $\mathrm{Z}(P) \subseteq U$. According to Algebra 1, $\mathrm{Z}(P) \neq 1$. Let $\overline{P} := P/\mathrm{Z}(P)$ and $\overline{U} := U/\mathrm{Z}(P)$. By induction, $\overline{U} < \mathrm{N}_{\overline{P}}(\overline{U})$. Thus there exists an $x \in P \setminus U$ with $xUx^{-1}/\mathrm{Z}(P) = \overline{x}\overline{U}\overline{x}^{-1} = \overline{U} = U/\mathrm{Z}(P)$. It follows that $x \in \mathrm{N}_P(U)$. □

Remark 8.8. The following theorem generalizes Theorem 5.8.

Theorem 8.9 (Wielandt)**.** *Let $N \trianglelefteq H \leq G$. Furthermore, let $H \cap xHx^{-1} \subseteq N$ for all $x \in G \setminus H$. Then there exists a normal subgroup M of G with $G = HM$ and $H \cap M = N$.*

Proof. If one denotes by π the set of prime divisors of $|H/N|$, then Theorem 8.3(i) is satisfied. Let $p \in \pi$, $Q \in \operatorname{Syl}_p(H)$ and $P \in \operatorname{Syl}_p(G)$ with $Q \subseteq P$. As is well known, $QN/N \in \operatorname{Syl}_p(H/N)$, so $QN/N \neq 1$ because of $p \bigm| |H/N|$. Consequently, $Q \nsubseteq N$. For $g \in \mathrm{N}_P(Q)$, $Q = gQg^{-1} \subseteq H \cap gHg^{-1}$. Because of $Q \nsubseteq N$, it follows that $g \in H$. Therefore $\mathrm{N}_P(Q) \subseteq H \cap P = Q$ and thus $P = Q$ according to Lemma 8.7. Consequently, $p \nmid |G : H|$, and Theorem 8.3(ii) is satisfied. Let $x, y \in H$ and $g \in G$ with $y = gxg^{-1}$. In the case $g \in H$, $yN = (gN)(xN)(gN)^{-1}$. So let $g \notin H$. Then $y = gxg^{-1} \in H \cap gHg^{-1} \subseteq N$ and analogously $x \in N$, so $yN = 1 = xN$ in H/N. Thus Theorem 8.3(iii) is satisfied. Finally, let $x \in H \setminus N$ and $g \in \mathrm{C}_G(x)$. Then $x = gxg^{-1} \in H \cap gHg^{-1}$, so $g \in H$. Consequently, $\mathrm{C}_G(x) \subseteq H$. Thus all assumptions of Theorem 8.3 are satisfied, and the assertion follows. □

Theorem 8.10. *Let $P \in \operatorname{Syl}_p(G)$. If any two elements in P that are conjugate in G are already conjugate in P, then P possesses a normal complement in G.*

Proof. Follows by Sylow and Theorem 8.6 with $\pi = \{p\}$ and $H = P$. □

Theorem 8.11 (Burnside's Transfer Theorem)**.** *Let $P \in \operatorname{Syl}_p(G)$ with $\mathrm{N}_G(P) = \mathrm{C}_G(P)$. Then P possesses a normal complement in G.*

Proof. Because of $P \subseteq \mathrm{N}_G(P) = \mathrm{C}_G(P)$, P is abelian. Let $x, y \in P$ and $g \in G$ with $gxg^{-1} = y$. Then $P \leq \mathrm{C}_G(x)$ and $g^{-1}Pg \leq g^{-1}\,\mathrm{C}_G(y)g = \mathrm{C}_G(g^{-1}yg) = \mathrm{C}_G(x)$. By Sylow, there exists a $c \in \mathrm{C}_G(x)$ with $cPc^{-1} = g^{-1}Pg$. Thus $gc \in \mathrm{N}_G(P) = \mathrm{C}_G(P) \leq \mathrm{C}_G(x)$. Hence $g \in \mathrm{C}_G(x)$ and $y = gxg^{-1} = x$. Now the assertion follows from Theorem 8.10. □

Remark 8.12. Usually, Theorem 8.11 is proven in group theory by means of transfer.

Theorem 8.13. *Let p be the smallest prime divisor of $|G|$, and let $P \in \operatorname{Syl}_p(G)$ be cyclic. Then P possesses a normal complement in G.*

Proof. As usual, $\mathrm{N}_G(P)$ acts by conjugation on P. This yields a homomorphism $\mathrm{N}_G(P) \to \operatorname{Aut}(P)$ with kernel $\mathrm{C}_G(P)$. Thus $\mathrm{N}_G(P)/\mathrm{C}_G(P)$ is isomorphic to a subgroup of $\operatorname{Aut}(P)$. Since P is cyclic, $P \cong \mathbb{Z}/p^n\mathbb{Z}$ for some $n \in \mathbb{N}$. In Algebra 1, it is shown that $\operatorname{Aut}(\mathbb{Z}/p^n\mathbb{Z}) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times$. This shows $|\mathrm{N}_G(P)/\mathrm{C}_G(P)| \mid |\operatorname{Aut}(P)| = \varphi(p^n) = p^{n-1}(p-1)$. Since P is cyclic, $P \subseteq \mathrm{C}_G(P)$ and $|\mathrm{N}_G(P)/\mathrm{C}_G(P)| \mid p-1$. Since p is the smallest prime divisor of $|G|$, even $\mathrm{N}_G(P) = \mathrm{C}_G(P)$ holds, and the assertion follows from Burnside's Transfer Theorem. □

Example 8.14. Let $P \in \operatorname{Syl}_2(G)$ be cyclic. According to Theorem 8.13, P has a normal complement N in G. According to Feit-Thompson, N is solvable. Because of $G/N = PN/N \cong P/P \cap N \cong P$, G is therefore also solvable. We prove a weaker statement without the theorem of Feit and Thompson.

Theorem 8.15. *If all Sylow groups of G are cyclic, then G is solvable.*

Proof. We argue by induction on $|G|$. wlog. let $G \neq 1$. Let p be the smallest prime divisor of G and $P \in \operatorname{Syl}_p(G)$. According to Theorem 8.13, there exists an $N \trianglelefteq G$ with $G = PN$ and $P \cap N = 1$. Every Sylow group of N lies in a Sylow group of G and is therefore cyclic. By induction, N is thus solvable. As is well known, every p-group is also solvable. With N and $G/N = PN/N \cong P/P \cap N \cong P$, G is therefore also solvable. □

Remark 8.16. In the situation of Theorem 8.15, one can further show that G' and G/G' are cyclic (without proof). In particular, $G'' = 1$.

Example 8.17. Groups of square-free order are solvable.

Theorem 8.18 (Frobenius)**.** *Let $P \in \mathrm{Syl}_p(G)$ and for every subgroup $1 \neq Q \leq P$ let $\mathrm{N}_G(Q)/\mathrm{C}_G(Q)$ be a p-group. Then P has a normal complement in G.*

Proof. According to Sylow, the assumption holds for all $P \in \mathrm{Syl}_p(G)$. Let Γ be the set of pairs (P, Q) with $P, Q \in \mathrm{Syl}_p(G)$, such that there exists a $c \in \mathrm{C}_G(P \cap Q)$ with $P = cQc^{-1}$. We show that Γ contains all pairs of Sylow groups. Let $P, P_1 \in \mathrm{Syl}_p(G)$ with $(P, P_1) \notin \Gamma$, such that $|P \cap P_1|$ is maximal. Obviously, then $D := P \cap P_1 < P$ (otherwise one could choose $c = 1$). Let $N := \mathrm{N}_G(D)$ and $P \cap N \subseteq S \in \mathrm{Syl}_p(N)$ as well as $P_1 \cap N \subseteq T \in \mathrm{Syl}_p(N)$. Finally, let $S \subseteq R \in \mathrm{Syl}_p(G)$. Since $S\,\mathrm{C}_G(D)/\mathrm{C}_G(D)$ is a p-Sylow group of $N/\mathrm{C}_G(D)$, the assumption implies $N = S\,\mathrm{C}_G(D)$. According to Sylow, there exists an $n \in N$ with $T = nSn^{-1}$. Because of $N = S\,\mathrm{C}_G(D) = \mathrm{C}_G(D)S$, we can assume $n \in \mathrm{C}_G(D)$. According to Lemma 8.7, it holds that

$$D < \mathrm{N}_P(D) = N \cap P \subseteq S \cap P \subseteq R \cap P.$$

By the choice of (P, P_1), there exists an $x \in \mathrm{C}_G(P \cap R) \subseteq \mathrm{C}_G(D)$ with $P = xRx^{-1}$. Analogously, it is also

$$D < \mathrm{N}_{P_1}(D) = N \cap P_1 \subseteq T \cap P_1 = nSn^{-1} \cap P_1 \subseteq nRn^{-1} \cap P_1$$

and there exists a $y \in \mathrm{C}_G(nRn^{-1} \cap P_1) \subseteq \mathrm{C}_G(D)$ with $nRn^{-1} = yP_1y^{-1}$. In total, we have $P = xRx^{-1} = xn^{-1}yP_1y^{-1}nx^{-1}$ with $xn^{-1}y \in \mathrm{C}_G(D) = \mathrm{C}_G(P \cap P_1)$. This contradiction shows that Γ contains all pairs (P, Q) with $P, Q \in \mathrm{Syl}_p(G)$.

Now let $x, y \in P$ and $g \in G$ with $y = gxg^{-1}$. Then $y \in P \cap gPg^{-1}$. According to what was just shown, there exists a $c \in \mathrm{C}_G(P \cap gPg^{-1}) \subseteq \mathrm{C}_G(y)$ with $cPc^{-1} = gPg^{-1}$. Since $P\,\mathrm{C}_G(P)/\mathrm{C}_G(P)$ is a p-Sylow group of $\mathrm{N}_G(P)/\mathrm{C}_G(P)$, it follows that $\mathrm{N}_G(P) = P\,\mathrm{C}_G(P)$ by assumption. Thus $c^{-1}g = ab$ with $a \in P$ and $b \in \mathrm{C}_G(P) \subseteq \mathrm{C}_G(x)$. Then $y = c^{-1}yc = c^{-1}gxg^{-1}c = abxb^{-1}a^{-1} = axa^{-1}$. The claim now follows from Theorem 8.10. $\square$

Theorem 8.19 (Focal Subgroup Theorem)**.** *For $P \in \mathrm{Syl}_p(G)$, we have*

$$P \cap G' = \langle xy^{-1} : x, y \in P \text{ are conjugate in } G \rangle.$$

Proof. If $x, y \in P$ are conjugate in G, then there exists a $g \in G$ with $xy^{-1} = xgx^{-1}g^{-1} = [x, g]$. This shows $P_1 := \langle xy^{-1} : x, y \in P \text{ are conjugate in } G \rangle \subseteq P \cap G'$. For $x, y \in P$, conversely, $[x, y] = x(yx^{-1}y^{-1}) = x(yxy^{-1})^{-1} \in P_1$. Thus $P' \subseteq P_1 \trianglelefteq P$ and P/P_1 is abelian. Let $\lambda \in \mathrm{Irr}(P/P_1) \subseteq \mathrm{Irr}(P)$. Let $g \in G$. Then there exists an $x \in G$ with $xg_px^{-1} \in P$. We define $\psi(g) := \lambda(xg_px^{-1})$. If also $yg_py^{-1} \in P$ for some $y \in G$, then

$$(xg_px^{-1})(yg_p^{-1}y^{-1}) = (xg_px^{-1})yx^{-1}(xg_px^{-1})^{-1}xy^{-1} \in P_1 \leq \mathrm{Ker}(\lambda).$$

Therefore $\lambda(xg_px^{-1}) = \lambda(yg_py^{-1})$ and ψ is well-defined. The same argument also shows $\psi \in \mathrm{CF}(G)$. We show $\psi \in \mathrm{Irr}(G)$ using Theorem 6.8. For this, let $E \leq G$ be elementary and $g, h \in E$. Since E is the direct product of its Sylow subgroups, $(gh)_p = g_ph_p$ holds and $\langle g_p, h_p \rangle$ is a p-group. Thus there exists an $x \in G$ with $x\langle g_p, h_p \rangle x^{-1} \leq P$. Then

$$\psi(gh) = \lambda(x(gh)_px^{-1}) = \lambda(xg_px^{-1}xh_px^{-1}) = \lambda(xg_px^{-1})\lambda(xh_px^{-1}) = \psi(g)\psi(h).$$

Because of $\psi(1) = \lambda(1) = 1$, we have $\psi_E \in \mathrm{Irr}(E)$. Because of $|\psi(g)| = 1$ for all $g \in G$, it also holds that $(\psi, \psi)_G = 1$. According to Theorem 6.8, $\psi \in \mathrm{Irr}(G)$. Now let $g \in P \setminus P_1$. By a suitable choice of λ, we can assume $\psi(g) = \lambda(g) \neq 1$. Because of $\psi(1) = 1$, it follows that $g \notin P \cap G' \leq G' \leq \mathrm{Ker}(\psi)$. $\square$

9 Zeros in the Character Table

Theorem 9.1 (Burnside)**.** *Let $\chi \in \operatorname{Irr}(G)$ with $\chi(1) > 1$. Then there exists a $g \in G$ with $\chi(g) = 0$.*

Proof. Let us assume $\chi(g) \neq 0$ for all $g \in G$. Let $\zeta := e^{2\pi i/|G|} \in \mathbb{C}$ and $\mathcal{G} := \operatorname{Gal}(\mathbb{Q}(\zeta)|\mathbb{Q})$. For $\gamma \in \mathcal{G}$ with $\gamma(\zeta) = \zeta^m$, it holds that $\gamma(\chi(g)) = \chi(g^m)$ according to Example 6.15. Because of $\gcd(|G|, m) = 1$, the map $g \mapsto g^m$ is a bijection on $G \setminus \{1\}$. Since $\mathcal{G}$ is abelian, $\gamma(\chi(g)\overline{\chi(g)}) = \chi(g^m)\overline{\chi(g^m)}$ holds. Thus $\omega := \prod_{1 \neq g \in G} |\chi(g)|^2 > 0$ is in the fixed field of $\mathcal{G}$, i.e. in $\mathbb{Q}$. On the other hand, ω is algebraic-integral and it follows that $\omega \in \mathbb{Z}$ from Lemma 3.5. In particular, $\omega \geq 1$ and the inequality of arithmetic and geometric means shows

$$\frac{1}{|G| - 1} \sum_{1 \neq g \in G} |\chi(g)|^2 \geq \omega^{\frac{1}{|G|-1}} \geq 1.$$

Thus

$$|G| = |G|(\chi, \chi)_G = \chi(1)^2 + \sum_{1 \neq g \in G} |\chi(g)|^2 \geq \chi(1)^2 + |G| - 1$$

and we obtain the contradiction $\chi(1) = 1$. □

Remark 9.2. For $\chi(1) = 1$, it obviously holds that $\chi(g) \neq 0$ for all $g \in G$. The next theorem shows where one can find zeros in the character table.

Theorem 9.3 (Brauer)**.** *Let $\chi \in \operatorname{Irr}(G)$ and $p \in \mathbb{P}$ with $p \nmid \frac{|G|}{\chi(1)}$. Then $\chi(g) = 0$ for all $g \in G$ with $p \mid |\langle g \rangle|$.*

Proof. We define a class function θ on G by

$$\theta(g) := \begin{cases} \chi(g) & \text{if } p \nmid |\langle g \rangle|, \\ 0 & \text{otherwise.} \end{cases}$$

We first show that θ is a virtual character. For this, let $E = P \times Q \leq G$ be elementary with $P \in \operatorname{Syl}_p(E)$ (note that E is not necessarily p-elementary). Then $\theta(x) = 0$ for all $x \in E \setminus Q$ and $\theta(x) = \chi(x)$ for $x \in Q$. For $\psi \in \operatorname{Irr}(E)$ we thus have

$$|P|(\theta_E, \psi)_E = \frac{1}{|Q|} \sum_{x \in Q} \chi(x)\psi(x^{-1}) = (\chi_Q, \psi_Q)_Q \in \mathbb{Z}.$$

Let $g \in K \in \operatorname{Cl}(G)$ and $\omega(g) := \omega_\chi(K) = \frac{|K|\chi(g)}{\chi(1)} = \frac{|G|\chi(g)}{|\mathrm{C}_G(g)|\chi(1)}$ (see Lemma 1.23). Then

$$|E|(\theta_E, \psi)_E = \sum_{x \in Q} \chi(x)\psi(x^{-1}) = \frac{\chi(1)}{|G|} \sum_{x \in Q} \omega(x)\psi(x^{-1})|\mathrm{C}_G(x)|.$$

For $x \in Q$, $P \subseteq \mathrm{C}_G(x)$ holds. Therefore

$$\frac{|G||Q|}{\chi(1)}(\theta_E, \psi)_E = \sum_{x \in Q} \omega(x)\psi(x^{-1})|\mathrm{C}_G(x) : P|$$

is an algebraic integer in $\mathbb{Q}$ according to Lemma 3.6. Consequently,

$$\frac{|G||Q|}{\chi(1)}(\theta_E, \psi)_E \in \mathbb{Z}.$$

Because of $\frac{|G||Q|}{\chi(1)} \in \mathbb{Z}$ and $\gcd(\frac{|G||Q|}{\chi(1)}, |P|) = 1$, it follows that $(\theta_E, \psi)_E \in \mathbb{Z}$. This shows that θ_E is a virtual character of E. By Brauer, θ is a virtual character of G (see proof of Theorem 6.8). In particular, $(\theta, \chi)_G \in \mathbb{Z}$. On the other hand,

$$0 < \frac{1}{|G|} \sum_{\substack{g \in G, \\ p \nmid |\langle g \rangle|}} \chi(g)\overline{\chi(g)} = (\theta, \chi)_G \leq (\chi, \chi)_G = 1.$$

Therefore $(\theta, \chi)_G = (\chi, \chi)_G = 1$ and

$$0 = (\chi - \theta, \chi)_G = \frac{1}{|G|} \sum_{\substack{g \in G, \\ p \mid |\langle g \rangle|}} |\chi(g)|^2,$$

i. e. $\chi(g) = 0$ for all $g \in G$ with $p \mid |\langle g \rangle|$. □

Remark 9.4. In the situation of Theorem 9.3, one says: χ has *p-defect* 0.

10 Finite linear groups

Remark 10.1. If G possesses a faithful representation of degree n, then G is isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{C})$. In this chapter, we will see that G possesses "large" abelian normal subgroups (in relation to n). In the case $n = 1$, G is even cyclic as a subgroup of $\mathbb{C}^\times$.

Definition 10.2. For $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ let

$$\|A\| := \sqrt{\mathrm{Trace}(A\overline{A}^{\mathrm{T}})} = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}.$$

Remark 10.3.

(i) If one writes $A \in \mathbb{C}^{n \times n}$ in the form $A = A_1 + A_2 i$ with $A_1, A_2 \in \mathbb{R}^{n \times n}$, then $\mathbb{C}^{n \times n}$ becomes a Euclidean space of dimension $2n^2$ (over $\mathbb{R}$) with the usual norm. In particular,

$$\|A + B\| \leq \|A\| + \|B\| \qquad \text{and} \qquad \|\alpha A\| = |\alpha|\|A\|$$

for $A, B \in \mathbb{C}^{n \times n}$ and $\alpha \in \mathbb{C}$.

(ii) Let $\mathrm{U}(n, \mathbb{C}) := \{A \in \mathrm{GL}(n, \mathbb{C}) : A^{-1} = \overline{A}^{\mathrm{T}}\} \leq \mathrm{GL}(n, \mathbb{C})$ be the *unitary group* of degree n. For $U, V \in \mathrm{U}(n, \mathbb{C})$ and $A \in \mathbb{C}^{n \times n}$, it then follows that

$$\begin{aligned}\|UAV\|^2 &= \mathrm{Trace}(UAV\overline{V}^{\mathrm{T}}\overline{A}^{\mathrm{T}}\overline{U}^{\mathrm{T}}) = \mathrm{Trace}(U(A\overline{A}^{\mathrm{T}})U^{-1}) = \mathrm{Trace}(U^{-1}UA\overline{A}^{\mathrm{T}}) \\ &= \mathrm{Trace}(A\overline{A}^{\mathrm{T}}) = \|A\|^2.\end{aligned}$$

(iii) The spectral theorem from linear algebra states that for every matrix $A \in \mathbb{C}^{n \times n}$ with $A\overline{A}^{\mathrm{T}} = \overline{A}^{\mathrm{T}}A$, there exists a matrix $U \in \mathrm{U}(n, \mathbb{C})$ such that UAU^{-1} is a diagonal matrix.[2] In particular, this holds if A is unitary or *Hermitian* (i. e. $\overline{A}^{\mathrm{T}} = A$).

[2] See lecture notes Linear Algebra

Theorem 10.4. *Every representation* $\Delta\colon G\to \mathrm{GL}(n,\mathbb{C})$ *is similar to a representation* $\Delta' : G \to \mathrm{U}(n,\mathbb{C})$.

Proof. The matrix $M := \sum_{g\in G}\Delta(g)\overline{\Delta(g)}^{\mathrm{T}}$ is Hermitian. By the spectral theorem, there exists $U \in \mathrm{U}(n,\mathbb{C})$ with $UMU^{-1} = D$, where D is a diagonal matrix. Because $\overline{U}^{\mathrm{T}} = U^{-1}$, it then follows that

$$D = UMU^{-1} = \sum_{g\in G} U\Delta(g)U^{-1}U\overline{\Delta(g)}^{\mathrm{T}}U^{-1} = \sum_{g\in G} U\Delta(g)U^{-1}\cdot\overline{U\Delta(g)U^{-1}}^{\mathrm{T}}.$$

Thus, we can assume that M is a diagonal matrix. By definition, the main diagonal entries of M are real and positive. In particular, $M = P^2$ for some $P \in \mathrm{GL}(n,\mathbb{R})$. As usual, $\Delta(h)M\overline{\Delta(h)}^{\mathrm{T}} = M$ and $(P^{-1}\Delta(h)P)(P\overline{\Delta(h)}^{\mathrm{T}}P^{-1}) = 1_n$ for all $h \in G$. Thus, $P^{-1}\Delta(h)P \in \mathrm{U}(n,\mathbb{C})$ for all $h \in G$. □

Theorem 10.5 (Jordan)**.** *There exists a function* $f : \mathbb{N}\to\mathbb{N}$ *with the following property: Every finite group* $G \le \mathrm{GL}(n,\mathbb{C})$ *possesses an abelian normal subgroup* $A \trianglelefteq G$ *with* $|G : A| \le f(n)$.

Proof (Tao). Induction on n: In the case $n = 1$, we can certainly set $f(1) := 1$. Let $n \ge 2$. The inclusion map $G \hookrightarrow \mathrm{GL}(n,\mathbb{C})$ is a representation of degree n. According to Theorem 10.4, we can assume $G \le \mathrm{U}(n,\mathbb{C})$ by replacing G with xGx^{-1} for some $x \in \mathrm{GL}(n,\mathbb{C})$.

Let $\epsilon > 0$ and

$$H := \langle x \in G : \|x-1\| < \epsilon\rangle.$$

For $x \in H$ with $\|x-1\| < \epsilon$ and $g \in G$, we have $\|gxg^{-1}-1\| = \|gxg^{-1}-gg^{-1}\| = \|g(x-1)g^{-1}\| = \|x-1\| < \epsilon$ according to Remark 10.3. Thus $H \trianglelefteq G$. Let $x_1,\dots,x_s \in G$ be a system of representatives for G/H. Then $\|x_i - x_j\| = \|x_ix_j^{-1}-1\| \ge \epsilon$ for $i \ne j$. On the other hand, $\|x_i\| = \sqrt{\mathrm{Trace}(x_i\overline{x_i}^{\mathrm{T}})} = \sqrt{\mathrm{Trace}(1_n)} = \sqrt{n}$ for $i = 1,\dots,s$. The subset $M := \{x \in \mathbb{C}^{n\times n} : \|x\| = \sqrt{n}\}$ of the Euclidean space $\mathbb{C}^{n\times n}$ is bounded and closed (i. e. compact). According to the Heine-Borel theorem from Analysis 1, M can be covered by finitely many, say $g(n)$, open balls of radius $\epsilon/2$. In each ball, at most one x_i can lie. In particular, $|G : H| = s \le g(n)$.

Assumption: $\mathrm{Z}(H) \le \mathbb{C}^\times 1_n$.
In the case $\mathrm{Z}(H) = H$, we are finished with $A := H$ and $f(n) := g(n)$. So let $x \in H \setminus \mathrm{Z}(H)$ such that $\|x-1\|$ is minimal (G finite). Since not all generators of H can lie in $\mathrm{Z}(H)$, we have $\|x-1\| < \epsilon$. Furthermore, let $y \in H$ with $\|y-1\| < \epsilon$. We choose $u \in \mathrm{U}(n,\mathbb{C})$ such that uxu^{-1} is a diagonal matrix (spectral theorem). Write $u(x-1)u^{-1} = \mathrm{diag}(\alpha_1,\dots,\alpha_n)$ and $u(y-1)u^{-1} = (\beta_{ij})$. Then

$$\begin{aligned}\|(x-1)(y-1)\|^2 &= \|u(x-1)u^{-1}u(y-1)u^{-1}\|^2 = \sum_{i,j}|\alpha_i\beta_{ij}|^2 \le \sum_{i,j,k}|\alpha_i|^2|\beta_{jk}|^2\\ &= \Big(\sum_i|\alpha_i|^2\Big)\Big(\sum_{j,k}|\beta_{jk}|^2\Big) = \|u(x-1)u^{-1}\|^2\|u(y-1)u^{-1}\|^2 = \|x-1\|^2\|y-1\|^2.\end{aligned}$$

Analogously, $\|(y-1)(x-1)\| \le \|x-1\|\|y-1\|$ also holds. It follows that

$$\|xyx^{-1}y^{-1}-1\| = \|xy-yx\| = \|(x-1)(y-1)-(y-1)(x-1)\| \le 2\|x-1\|\|y-1\| < 2\epsilon\|x-1\| < \|x-1\|$$

for $\epsilon < \frac{1}{2}$. By the choice of x, it follows that $z := xyx^{-1}y^{-1} \in \mathrm{Z}(H)$. According to the assumption, $z = \lambda 1_n$ for some $\lambda \in \mathbb{C}$. Obviously $\det(z) = \det(xyx^{-1}y^{-1}) = 1$, i. e. $\lambda^n = 1$. In particular, there are only n possibilities for λ. We have $|\lambda-1|\sqrt{n} = \|z-1\| < \|x-1\| < \epsilon$. If one chooses ϵ small enough, it follows that $\lambda = 1$ and $xyx^{-1}y^{-1} = z = 1$. Thus x commutes with all generators of H. This yields the contradiction $x \in \mathrm{Z}(H)$.

Thus there exists $z \in \mathrm{Z}(H) \setminus \mathbb{C}1_n$. Because $z \in \mathrm{U}(n, \mathbb{C})$, there exists a $u \in \mathrm{U}(n, \mathbb{C})$ with

$$d := uzu^{-1} = \begin{pmatrix} \alpha_1 1_{n_1} & & \\ & \ddots & \\ & & \alpha_k 1_{n_k} \end{pmatrix},$$

where $\alpha_1, \ldots, \alpha_k$ are pairwise distinct (wlog.). Because $z \notin \mathbb{C}1_n$, we have $k > 1$. One easily shows:

$$\mathrm{C}_{\mathrm{GL}(n,\mathbb{C})}(d) = \begin{pmatrix} \mathrm{GL}(n_1, \mathbb{C}) & & 0 \\ & \ddots & \\ 0 & & \mathrm{GL}(n_k, \mathbb{C}) \end{pmatrix} \cong \mathrm{GL}(n_1, \mathbb{C}) \times \ldots \times \mathrm{GL}(n_k, \mathbb{C}).$$

Thus also $H \le \mathrm{C}_{\mathrm{GL}(n,\mathbb{C})}(z) = u^{-1}\,\mathrm{C}_{\mathrm{GL}(n,\mathbb{C})}(d)u \cong \mathrm{GL}(n_1, \mathbb{C}) \times \ldots \times \mathrm{GL}(n_k, \mathbb{C})$. Let $\pi_i : H \to \mathrm{GL}(n_i, \mathbb{C})$ be the i-th projection. Because $k > 1$, we have $n_i < n$. By induction, there exists an abelian normal subgroup $N_i/\operatorname{Ker}(\pi_i) \trianglelefteq H/\operatorname{Ker}(\pi_i) \cong \pi_i(H) \le \mathrm{GL}(n_i, \mathbb{C})$ with $|H : N_i| = |H/\operatorname{Ker}(\pi_i) : N_i/\operatorname{Ker}(\pi_i)| \le f(n_i)$. Let

$$\pi : H \to H/\operatorname{Ker}(\pi_1) \times \ldots \times H/\operatorname{Ker}(\pi_k), \qquad g \mapsto (g\operatorname{Ker}(\pi_1), \ldots, g\operatorname{Ker}(\pi_k)).$$

Then $\operatorname{Ker}(\pi) = \bigcap_{i=1}^{k} \operatorname{Ker}(\pi_i) = 1$. We define

$$K := \pi^{-1}(N_1/\operatorname{Ker}(\pi_1) \times \ldots \times N_k/\operatorname{Ker}(\pi_k)) = N_1 \cap \ldots \cap N_k.$$

Since π is injective, K is an abelian normal subgroup of H. Since the map $H/K \to H/N_1 \times \ldots \times H/N_k$, $gK \mapsto (gN_1, \ldots, gN_k)$ is also injective, we have

$$|H : K| \le |H : N_1| \ldots |H : N_k| \le f(n_1) \ldots f(n_k).$$

The function $h(n) := \max\{f(i) : 1 \le i \le n-1\}^n$ thus satisfies $|H : K| \le h(n)$. In total, $|G : K| = |G : H||H : K| \le g(n)h(n)$. Since the normal subgroup relation is not transitive, K is not necessarily normal in G. We therefore consider the action $\varphi : G \to \mathrm{Sym}(G/K)$ with ${}^{g}(hK) := ghK$ for $g, h \in G$ (cf. proof of Theorem 7.11). One easily shows that $A := \operatorname{Ker}(\varphi) \subseteq K$ holds. In particular, A is an abelian normal subgroup of G with $|G : A| = |G : \operatorname{Ker}(\varphi)| \le |\mathrm{Sym}(G/K)| = |G : K|! \le (g(n)h(n))!$. The function $f(n) := (g(n)h(n))!$ thus satisfies the claim. □

Remark 10.6.

(i) As in Exercise 20, one shows that $G := \mathrm{SL}(2, 5)$ acts on the set of the six one-dimensional subspaces of $\mathbb{F}_5^2$ with kernel $\mathrm{Z}(G) = \langle -1_2 \rangle$. It follows that $\mathrm{SL}(2, 5)/\mathrm{Z}(G) \cong A_5$ and $\mathrm{Z}(G)$ is the largest abelian normal subgroup of G. Furthermore, G possesses a faithful (irreducible) character of degree 2. This shows $f(2) \ge |G : \mathrm{Z}(G)| = 60$. In fact, equality holds (without proof).

(ii) Collins has shown $|G : A) \le (n+1)!$ for $n \ge 71$ in the situation of Theorem 10.5. According to Exercise 30, this bound is also optimal. The proof, however, uses the classification of finite simple groups.

Definition 10.7. A subgroup $H \le G$ is called a π-*Hall subgroup*, if H is a π-group and no prime divisor of $|G : H|$ lies in π. In particular, $\gcd(|H|, |G : H|) = 1$.

Example 10.8.

(i) The p-Hall subgroups of G are exactly the p-Sylow subgroups.

(ii) Let G be abelian and $S_p \in \mathrm{Syl}_p(G)$. For $\pi \subseteq \mathbb{P}$, then $G_\pi := \prod_{p\in\pi} S_p$ is a π-Hall subgroup of G.

(iii) A_5 possesses no $\{2,5\}$-Hall subgroup.

(iv) According to Exercise 16, Frobenius kernels and Frobenius complements are always Hall subgroups.

Definition 10.9.

(i) For $n \in \mathbb{N}$ let $\mathbb{Q}_n := \mathbb{Q}(e^{2\pi i/n})$.

(ii) Let $p \in \mathbb{P}$ and $|G| = mp^a$ with $p \nmid m$. A character χ of G is called *p-rational*, if $\chi(g) \in \mathbb{Q}_m$ holds for all $g \in G$.

Example 10.10. In the case $p \nmid |\langle g\rangle|$, $\chi(g) \in \mathbb{Q}_m$ (as a sum of $|\langle g\rangle|$-th roots of unity). In particular, every character is p-rational if $p \nmid |G|$.

Lemma 10.11. *Let $p \neq q$ be prime numbers such that G has no element of order pq. Then every character $\chi \in \mathrm{Irr}(G)$ is p-rational or q-rational.*

Proof. Suppose that $\chi \in \mathrm{Irr}(G)$ is neither p-rational nor q-rational. We write $n := |G| = p^a m_p = q^b m_q$ with $p \nmid m_p$ and $q \nmid m_q$. Let $\mathcal{G}_p := \mathrm{Gal}(\mathbb{Q}_n|\mathbb{Q}_{m_p})$ and $\mathcal{G}_q := \mathrm{Gal}(\mathbb{Q}_n|\mathbb{Q}_{m_q})$. Then there exist $\gamma_p \in \mathcal{G}_p$ and $\gamma_q \in \mathcal{G}_q$ with ${}^{\gamma_p}\chi \neq \chi \neq {}^{\gamma_q}\chi$ (see Example 6.15).

If $g \in G$ with $p \nmid |\langle g\rangle|$, then $\chi(g) \in \mathbb{Q}_{m_p}$. In particular, $\gamma_p(\chi(g)) = \chi(g)$. Analogously, $\gamma_q(\chi(g)) = \chi(g)$ for $q \nmid |\langle g\rangle|$. Since G has no element of order pq, it holds that $p \nmid |\langle g\rangle|$ or $q \nmid |\langle g\rangle|$ for all $g \in G$. This shows

$$((\chi - {}^{\gamma_p}\chi), (\chi - {}^{\gamma_q}\chi))_G = \frac{1}{|G|}\sum_{g\in G}(\chi - {}^{\gamma_p}\chi)(g)\overline{(\chi - {}^{\gamma_q}\chi)(g)} = 0.$$

Because ${}^{\gamma_p}\chi, {}^{\gamma_q}\chi \in \mathrm{Irr}(G)$, we have $(\chi, {}^{\gamma_p}\chi)_G = (\chi, {}^{\gamma_q}\chi)_G = 0$. The contradiction follows:

$$0 = (\chi,\chi)_G + ({}^{\gamma_p}\chi, {}^{\gamma_q}\chi)_G \geq 1.$$

□

Lemma 10.12. *For $\gcd(n,m) = 1$, we have $\mathbb{Q}_n \cap \mathbb{Q}_m = \mathbb{Q}$.*

Proof. Let $\zeta := e^{\frac{2\pi i}{nm}}$. Because $\mathbb{Q}_n = \mathbb{Q}(\zeta^m)$ and $\mathbb{Q}_m = \mathbb{Q}(\zeta^n)$, the restriction map

$$\Gamma\colon\ \mathrm{Gal}(\mathbb{Q}_{nm}|\mathbb{Q}_n) \to \mathrm{Gal}(\mathbb{Q}_m|\mathbb{Q}_n \cap \mathbb{Q}_m)$$

is a well-defined homomorphism. Let $\gamma \in \mathrm{Ker}(\Gamma)$. Because $\gcd(n,m) = 1$, there exist $a, b \in \mathbb{Z}$ with $1 = an + bm$. Then $\gamma(\zeta) = \gamma(\zeta^{an+bm}) = \gamma(\zeta^n)^a\gamma(\zeta^m)^b = \zeta^{na}\zeta^{mb} = \zeta$. Thus Γ is injective and

$$\varphi(m) = \frac{\varphi(nm)}{\varphi(n)} = \frac{[\mathbb{Q}_{nm} : \mathbb{Q}]}{[\mathbb{Q}_n : \mathbb{Q}]} = [\mathbb{Q}_{nm} : \mathbb{Q}_n] = |\mathrm{Gal}(\mathbb{Q}_{nm}|\mathbb{Q}_n)| \leq |\mathrm{Gal}(\mathbb{Q}_m|\mathbb{Q}_n \cap \mathbb{Q}_m)| \leq \varphi(m).$$

The fundamental theorem of Galois theory implies the assertion. □

Lemma 10.13. *Let χ be a faithful p-rational character of G, where $p \mid |G|$. Then $\chi(1) \geq p - 1$.*

Proof. wlog. let $p \geq 3$. We write $|G| = mp^a$ with $p \nmid m$. By assumption, $\chi(g) \in \mathbb{Q}_m$ for all $g \in G$. Let $P \leq G$ with $|P| = p$. Then χ_P has values in $\mathbb{Q}_p$. According to Lemma 10.12, $\chi(x) \in \mathbb{Q}$ for $x \in P$. Since χ is faithful, χ_P contains a non-trivial constituent $\psi \in \mathrm{Irr}(P)$. For $1 \neq \gamma \in \mathrm{Gal}(\mathbb{Q}_p|\mathbb{Q})$, ${}^\gamma\psi \neq \psi$ is an irreducible constituent of ${}^\gamma(\chi_P) = \chi_P$. Because $|\mathrm{Gal}(\mathbb{Q}_p|\mathbb{Q})| = p - 1$, χ_P thus possesses at least $p - 1$ irreducible constituents. This shows the assertion. □

Lemma 10.14 (Frattini Argument)**.** *Let $N \trianglelefteq G$ and $P \in \mathrm{Syl}_p(N)$. Then $G = N\,\mathrm{N}_G(P)$.*

Proof. For $g \in G$, $gPg^{-1} \leq gNg^{-1} = N$. By Sylow, there exists an $x \in N$ with $gPg^{-1} = xPx^{-1}$. Thus $g = x(x^{-1}g) \in N\,\mathrm{N}_G(P)$. □

Lemma 10.15. *Let G be abelian and $p \bigm| |G|$. Then there exists a subgroup $H \leq G$ with $|G : H| = p$.*

Proof. Since G is the direct product of its Sylow subgroups, we can assume that G is a p-group. In the case $|G| = p$, we can choose $H = 1$. Now let $|G| > p$, and let $x \in G$ be an element of order p. By induction, $G/\langle x\rangle$ possesses a subgroup $H/\langle x\rangle$ with $|G : H| = |G/\langle x\rangle : H/\langle x\rangle| = p$. □

Theorem 10.16 (Blichfeldt)**.** *Every finite group $G \leq \mathrm{GL}(n, \mathbb{C})$ possesses an abelian π-Hall subgroup for $\pi := \{p \in \mathbb{P} : p > n + 1\}$.*

Proof. We argue by double induction on n and then on $|G|$. In the case $n = 1$, G is abelian and the claim follows from Example 10.8(ii). So let $n \geq 2$. Let χ be the faithful character of degree n arising from the embedding $G \hookrightarrow \mathrm{GL}(n, \mathbb{C})$.

Claim: Every π-subgroup $H \leq G$ is abelian.
Let ψ be an irreducible constituent of χ_H. Then $\psi(1) \bigm| |H|$. Every prime divisor p of $\psi(1)$ thus satisfies $n + 1 < p \leq \psi(1) \leq \chi(1) = n$. Consequently, $\psi(1) = 1$. Thus χ_H is a sum of irreducible characters $\psi_1, \ldots, \psi_k$ of degree 1. Hence

$$H' \subseteq \mathrm{Ker}(\psi_1) \cap \ldots \cap \mathrm{Ker}(\psi_k) = \mathrm{Ker}(\chi_H) = \mathrm{Ker}(\chi) \cap H = 1$$

(see Exercise 9) and H is abelian.

It therefore suffices to show that some π-Hall subgroup exists.

Claim: $\chi \in \mathrm{Irr}(G)$.
Let χ be reducible and $\chi = \chi_1 + \chi_2$ with $K_i := \mathrm{Ker}(\chi_i)$ for $i = 1, 2$. Then $G/K_i \leq \mathrm{GL}(\chi_i(1), \mathbb{C})$ with $\chi_i(1) < n$. By induction, G/K_i possesses abelian π_i-Hall subgroups with $\pi_i := \{p \in \mathbb{P} : p > \chi_i(1) + 1\}$ for $i = 1, 2$. Because of $\pi \subseteq \pi_i$, there also exist π-Hall subgroups H_i/K_i of G/K_i for $i = 1, 2$ (Example 10.8(ii)). If $H_i < G$ for some i, then by induction there exists a π-Hall subgroup H of H_i. For all $p \in \pi$, we then have $p \nmid |G/K_i : H_i/K_i||H_i : H| = |G : H_i||H_i : H| = |G : H|$. We can therefore assume $H_i = G$ for $i = 1, 2$. In particular, G/K_i is abelian. Because of $K_1 \cap K_2 = \mathrm{Ker}(\chi_1) \cap \mathrm{Ker}(\chi_2) = \mathrm{Ker}(\chi) = 1$, the map $G \to G/K_1 \times G/K_2$, $g \mapsto (gK_1, gK_2)$ is injective. In particular, G is also abelian and the claim follows from Example 10.8(ii).

Claim: $G' = G$.
Let $G' < G$. By Lemma 10.15, there exists a normal subgroup $N \trianglelefteq G$ with $|G : N| = p \in \mathbb{P}$ (since G/G' is abelian). By induction, N possesses a π-Hall subgroup H. In the case $p \notin \pi$, H is also a π-Hall subgroup of G. So let $p \in \pi$. In the case $H \trianglelefteq G$, HP is a π-Hall subgroup of G, where $P \in \mathrm{Syl}_p(G)$. So let $H \ntrianglelefteq G$. Then there exists a Sylow subgroup Q of H with $Q \ntrianglelefteq G$ (otherwise H, as a product

of normal subgroups, would also be normal). Then Q is also a Sylow subgroup of N and the Frattini argument shows $G = N\,\mathrm{N}_G(Q)$. Since H is abelian, $H \subseteq \mathrm{N}_G(Q)$ holds. Furthermore,

$$|G : \mathrm{N}_G(Q)| = |N\,\mathrm{N}_G(Q) : \mathrm{N}_G(Q)| = |N : N \cap \mathrm{N}_G(Q)| = |N : \mathrm{N}_N(Q)| \;\big|\; |N : H|.$$

In particular, the prime divisors of $|G : \mathrm{N}_G(Q)|$ do not lie in π. Because $\mathrm{N}_G(Q) < G$, $\mathrm{N}_G(Q)$ possesses a π-Hall subgroup K. Obviously, K is then also a π-Hall subgroup of G and we are finished.

Claim: $\mathrm{Z}(G)$ is a π'-group.
Let $Z \le \mathrm{Z}(G)$ with $|Z| = p \in \pi$. By Theorem 2.14, $Z \subseteq \mathrm{Z}(\chi)$. Let Δ be a representation with character χ. For $1 \ne x \in Z$, we then have $\Delta(x) = \lambda 1_n$ for a p-th root of unity $\lambda \ne 1$. Because of $p > n$, $\det(\Delta(x)) = \lambda^n \ne 1$. Thus $\det \Delta \ne 1_G$ in contradiction to $G' = G$.

Now let $H \le G$ be a π-subgroup of maximal order.

Claim: $\gcd(|H|, |G : H|) = 1$.
Let $1 \ne x \in H$ be a p-element. Since $\mathrm{Z}(H)$ is a π-group, $C := \mathrm{C}_G(x) < G$ holds. By induction, C possesses a π-Hall subgroup K. Since H is abelian (see above), $H \subseteq C$ holds and thus $|H| \le |K|$. From the maximality of H, it follows that $|H| = |K|$. Let $x \in P \in \mathrm{Syl}_p(G)$. Since P, as a π-subgroup, is abelian, $P \le C$ and $p \nmid |G : C|$ hold. On the other hand, $p \nmid |C : K| = |C : H|$ and $p \nmid |G : C||C : H| = |G : H|$.

We may assume that G possesses at least one π-element, i.e., $H \ne 1$ (otherwise G is a π'-group and the claim holds with $H = 1$). So let p be a prime divisor of $|H|$. We assume that H is not a π-Hall subgroup. Thus there exists a $q \in \pi$ with $q \mid |G : H|$.

Claim: G possesses no element of order pq.
Otherwise there exist $x, y \in G$ with $|\langle x \rangle| = p$, $|\langle y \rangle| = q$ and $y \in \mathrm{C}_G(x)$. By Sylow, we may assume $x \in H$. With the above notation, $y \in C$, $q \mid |C|$ and therefore $q \mid |K| = |H|$. This contradicts $\gcd(|H|, |G : H|) = 1$.

By Lemma 10.11, χ is r-rational for an $r \in \{p, q\}$. Lemma 10.13 now implies $n = \chi(1) \ge r - 1$ in contradiction to $r \in \pi$. □

Theorem 10.17 (Wielandt). *For $\pi \subseteq \mathbb{P}$, any two abelian π-Hall subgroups of G are conjugate.*

Proof. Let $A, B \le G$ be abelian π-Hall subgroups. We argue by induction on $|\pi|$. In the case $|\pi| \le 1$, the assertion follows from Sylow's Theorem. Let $|\pi| \ge 2$. We write $A = \prod_{p \in \pi} S_p$ and $B = \prod_{p \in \pi} T_p$ with $S_p, T_p \in \mathrm{Syl}_p(G)$. Let $q \in \pi$ be fixed. By induction, there exists a $g \in G$ with

$$g\Big(\prod_{q \ne p \in \pi} S_p\Big)g^{-1} = \prod_{q \ne p \in \pi} T_p.$$

Replacing A by gAg^{-1}, one can assume $S_p = T_p$ for all $p \ne q$. Obviously, then $S_q, T_q \in \mathrm{Syl}_q(\mathrm{N}_G(\prod_{q \ne p \in \pi} S_p))$. By Sylow, there exists an $h \in \mathrm{N}_G(\prod_{q \ne p \in \pi} S_p)$ with $hS_qh^{-1} = T_q$. Then

$$hAh^{-1} = hS_qh^{-1}h\Big(\prod_{q \ne p \in \pi} S_p\Big)h^{-1} = T_q \prod_{q \ne p \in \pi} S_p = B.$$ □

Example 10.18. In general, two π-Hall subgroups of G are not conjugate. Let for example $G := \mathrm{GL}(3,2)$ and

$$H := \begin{pmatrix} 1 & * \\ 0 & \mathrm{GL}(2,2) \end{pmatrix} \le G, \qquad\qquad K := \begin{pmatrix} \mathrm{GL}(2,2) & * \\ 0 & 1 \end{pmatrix} \le G.$$

Because $|G| = 2^3 \cdot 3 \cdot 7$ and $|H| = |K| = 24$, H and K are $\{2,3\}$-Hall subgroups of G. Suppose that there exists a $g \in G$ with $gHg^{-1} = K$. Let $e_1 := (1,0,0)$ and $g \cdot e_1 = (\alpha, \beta, \gamma) \in \mathbb{F}_2^3$. For all $x \in K$, it then follows that

$$x(\alpha, \beta, \gamma) = g \underbrace{(g^{-1}xg)}_{\in H} e_1 = g \cdot e_1 = (\alpha, \beta, \gamma).$$

With $\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right) \in K$, it follows that $\beta = \gamma = 0$. From $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \in K$, the contradiction $\alpha = 0$ now follows. Thus H and K are not conjugate in G.

Theorem 10.19 (Brauer-Burnside)**.** *Let θ be a faithful character of G that takes exactly r distinct values $a_1, \ldots, a_r$. Then every irreducible character of G occurs among the irreducible constituents of $\theta^0 = 1_G, \theta, \theta^2, \ldots, \theta^{r-1}$.*

Proof. Let $\chi \in \mathrm{Irr}(G)$ be no irreducible constituent of $\theta^0, \theta, \ldots, \theta^{r-1}$, and let $A_j := \{g \in G : \theta(g) = a_j\}$ for $j = 1, \ldots, r$. Then

$$0 = |G|(\theta^s, \chi)_G = \sum_{g \in G} \theta^s(g)\chi(g^{-1}) = \sum_{j=1}^{r} a_j^s \sum_{g \in A_j} \chi(g^{-1})$$

for $s = 0, \ldots, r-1$. Therefore $\left(\sum_{g \in A_1} \chi(g^{-1}), \ldots, \sum_{g \in A_r} \chi(g^{-1})\right)$ is a solution of the homogeneous system of equations with the coefficient matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_r \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{r-1} & a_2^{r-1} & \cdots & a_r^{r-1} \end{pmatrix}.$$

Matrices of this form (Vandermonde matrix) are known to be invertible. Consequently, $\sum_{g \in A_j} \chi(g^{-1}) = 0$ for $j = 1, \ldots, r$. Let $j \in \{1, \ldots, r\}$ with $1 \in A_j$, so $a_j = \theta(1)$. Since θ is faithful, $A_j = \{1\}$, and one has the contradiction $\chi(1) = 0$. □

Example 10.20. For the regular character θ of G, $r = 2$ holds in Theorem 10.19 (if $G \neq 1$). Indeed, we already know that every irreducible character occurs in $\theta = \theta^{r-1}$.

11 The characters of S_n and A_n

Remark 11.1.

(i) According to Exercise 27, two elements in S_n are conjugate if and only if they have the same cycle type. Consequently, one can identify the set of conjugacy classes with the partitions of n. Here, a *partition* of n is a sequence of natural numbers $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_k \ge 1$ and $\lambda_1 + \ldots + \lambda_k = n$.

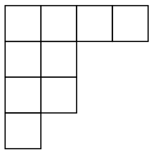
(ii) In the following, we will calculate the character table of S_n by also identifying the irreducible characters with the partitions. According to Exercise 27, this is an integer matrix.

Definition 11.2.

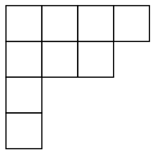
(i) Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n. The *Young diagram* of λ is an arrangement of n boxes with λ_i boxes in the i-th row. By reflection across the diagonal, one obtains the *conjugate* Young diagram with partition $\lambda' = (\lambda'_1, \ldots, \lambda'_l)$ with $\lambda'_i := |\{j : \lambda_j \geq i\}|$ for $i = 1, \ldots, l$. Certainly $\lambda'' = \lambda$ (cf. Example 11.3).

(ii) A *Young tableau* (of λ) is a Young diagram (of λ) filled with the numbers $1, \ldots, n$, where in each row the numbers are sorted in ascending order.

Example 11.3. Let $\lambda = (4, 2, 2, 1) = (4, 2^2, 1)$ be a partition of 9. Then the Young diagram of λ, a Young tableau and the conjugate Young diagram are given by:



2	3	6	7
1	8		
5	9		
4			



Remark 11.4.

(i) We will often identify the Young tableaux Y of $\lambda = (\lambda_1, \ldots, \lambda_k)$ with the set partitions $Y = (Y_1, \ldots, Y_k)$ with $Y_1 \cup \ldots \cup Y_k = \{1, \ldots, n\}$ and $|Y_i| = \lambda_i$ for $i = 1, \ldots, k$. Obviously, S_n then acts transitively on the set of Young tableaux of λ via ${}^gY := (g(Y_1), \ldots, g(Y_k))$. Let φ_λ be the corresponding permutation character (see Exercise 29). The stabilizer of Y is given by the *Young subgroup*

$$S_Y := (S_n)_Y = \mathrm{Sym}(Y_1) \times \ldots \times \mathrm{Sym}(Y_k) \cong S_{\lambda_1} \times \ldots \times S_{\lambda_k}.$$

According to Exercise 29, $\boxed{\varphi_\lambda = 1_{S_Y}^{S_n}.}$ Let sgn be the alternating character of S_n (see Example 1.2). We set

$$\boxed{\psi_\lambda := \mathrm{sgn} \cdot \varphi_\lambda = (\mathrm{sgn}_{S_Y} \cdot 1_{S_Y})^{S_n} = (\mathrm{sgn}_{S_Y})^{S_n}.}$$

(ii) Let $Y' = (Y'_1, \ldots, Y'_l)$ be the Young tableau conjugate to Y (as a set partition). Then $|Y_i \cap Y'_j| \leq 1$ for all i, j (intersection of row and column). This shows $S_Y \cap S_{Y'} = 1$.

Example 11.5.

(i) For $\lambda = (n)$, $S_Y = S_n$, $\varphi_\lambda = 1_{S_n}$ and $\psi_\lambda = \mathrm{sgn}$.

(ii) For $\lambda = (1^n)$, $S_Y = 1$, and $\varphi_\lambda = 1_1^{S_n} = (\mathrm{sgn}_1)^{S_n} = \psi_\lambda$ is the regular character of S_n.

(iii) For $\lambda = (n-1, 1)$, the set of Young tableaux can be identified with $\{1, \ldots, n\}$. Thus φ_λ is the natural permutation character of degree n (Exercise 29).

Theorem 11.6. *For every partition λ of n, $(\varphi_\lambda, \psi_{\lambda'})_{S_n} = 1$. In particular, there is exactly one common irreducible constituent χ_λ of φ_λ and $\psi_{\lambda'}$.*

Proof. Let Y be a Young tableau of λ. According to Frobenius and Mackey,

$$\begin{aligned}(\varphi_\lambda, \psi_{\lambda'})_{S_n} &= (1_{S_Y}^{S_n}, (\mathrm{sgn}_{S_{Y'}})^{S_n})_{S_n} = ((1_{S_Y}^{S_n})_{S_{Y'}}, \mathrm{sgn}_{S_{Y'}})_{S_{Y'}} \\ &= \sum_{S_{Y'}gS_Y \in S_{Y'}\backslash S_n/S_Y} (1_{S_{Y'}\cap gS_Yg^{-1}}^{S_{Y'}}, \mathrm{sgn}_{S_{Y'}})_{S_{Y'}} = \sum_{S_{Y'}gS_Y \in S_{Y'}\backslash S_n/S_Y} (1_{D_g}, \mathrm{sgn}_{D_g})_{D_g} \qquad (11.1)\end{aligned}$$

with $D_g := S_{Y'} \cap gS_Yg^{-1} = S_{Y'} \cap S_{{}^gY}$. For $g = 1$, we have $D_g = 1$ and $(1_{D_g}, \mathrm{sgn}_{D_g})_{D_g} = 1$ according to Remark 11.4(ii). We must show that all other summands in (11.1) vanish. So let $(1_{D_g}, \mathrm{sgn}_{D_g})_{D_g} > 0$ for some $g \in S_n$. Suppose two numbers a, b lie in the same row of ${}^gY$ and in the same column of Y. Then the transposition (a, b) would be in D_g and $1_{D_g} \neq \mathrm{sgn}_{D_g}$. Therefore, the numbers of a row of ${}^gY$ are distributed over pairwise distinct columns of Y. In particular, there exists an $h_1 \in S_{Y'}$ such that the first rows of ${}^gY$ and ${}^{h_1}Y$ coincide as sets. Analogously, there exists $h_2 \in S_{Y'}$ such that the first two rows of ${}^gY$ and ${}^{h_2h_1}Y$ coincide, and so on. Finally, there exists $h \in S_{Y'}$ with ${}^gY = {}^hY$, i. e. $h^{-1}g \in S_Y$. This shows $S_{Y'}gS_Y = S_{Y'}S_Y$ and we are finished. □

Definition 11.7. Let $\leq$ be the lexicographic order on the set of partitions of n, i. e.

$$\lambda < \mu \;:\Longleftrightarrow\; \exists k : \lambda_1 = \mu_1, \ldots, \lambda_{k-1} = \mu_{k-1}, \lambda_k < \mu_k.$$

Theorem 11.8 (Frobenius-Young). *For all $n \in \mathbb{N}$, $\mathrm{Irr}(S_n) = \{\chi_\lambda : \lambda$ partition of $n\}$.*

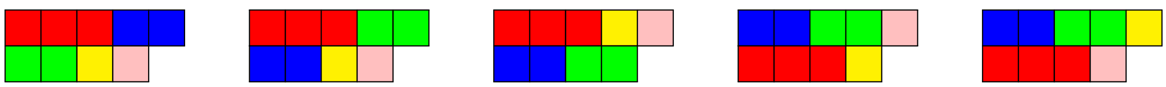
Proof. It suffices to show that the χ_λ are pairwise distinct. So let $\chi_\lambda = \chi_\mu$ for partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$ of n. By definition, $(\varphi_\lambda, \psi_{\mu'})_{S_n} \neq 0$. Let Y and Z be Young tableaux of λ and μ, respectively. As in the proof of Theorem 11.6, there exists a $g \in S_n$ with $S_{Z'} \cap S_{{}^gY} = 1$. Replacing Y by ${}^gY$, one can assume $g = 1$. With $Y = (Y_1, \ldots, Y_k)$ and $Z' = (Z'_1, \ldots, Z'_m)$, it thus holds that $|Y_i \cap Z'_j| \leq 1$. We want to show $\lambda \leq \mu$. It holds that $\lambda_1 = |Y_1| \leq m = \mu_1$. Now assume that $\lambda_j = \mu_j$ for $j \leq i$. One can then distribute the first i rows of Y exactly onto the first i columns of Z'. Because $Y_{i+1} \subseteq \{1, \ldots, n\} = Z'_1 \cup \ldots \cup Z'_m$, Z' thus has at least $|Y_{i+1}| = \lambda_{i+1}$ rows with at least $i + 1$ boxes, i. e. $\mu_{i+1} = |\{j : |Z'_j| \geq i + 1\}| \geq \lambda_{i+1}$. This shows $\lambda \leq \mu$. Because $(\varphi_\mu, \psi_{\lambda'})_{S_n} \neq 0$, analogously $\mu \leq \lambda$ also holds and we are finished. □

Lemma 11.9. *For a partition λ, $\varphi_\lambda = \sum_{\mu \geq \lambda} (\varphi_\lambda, \chi_\mu)_{S_n} \chi_\mu$.*

Proof. Let $(\varphi_\lambda, \chi_\mu)_{S_n} \neq 0$ for a partition μ. Then also $(\varphi_\lambda, \psi_{\mu'})_{S_n} \neq 0$ and as in the proof of Theorem 11.8 it follows that $\lambda \leq \mu$. □

Lemma 11.10. *Let λ and μ be partitions of n. For an element $g \in S_n$ of cycle type μ, $\varphi_\lambda(g)$ is then the number of possibilities to distribute the components of μ into the components of λ.*

Example 11.11. Let $\lambda = (5, 4)$ and $\mu = (3, 2^2, 1^2)$. Then $\varphi_\lambda(g) = 5$, illustrated by colored Young diagrams:



Proof of Lemma 11.10. Let $\mu = (\mu_1, \ldots, \mu_k)$ and $g = \sigma_1 \ldots \sigma_k$ with disjoint cycles σ_i of length μ_i for $i = 1, \ldots, k$. According to Exercise 29, $\varphi_\lambda(g)$ is the number of Young tableaux of λ that remain fixed by g. A Young tableau Y of λ remains fixed under g if and only if for $i = 1, \ldots, k$ the non-trivial orbit of σ_i lies in one Y_j. For Y there are therefore exactly as many possibilities as there are possibilities to distribute the μ_i into the Young diagram of λ. □

Remark 11.12. Lemma 11.9 and Lemma 11.10 allow to calculate $\mathrm{Irr}(S_n)$ recursively. According to Example 11.5, $\chi_{(n)} = \varphi_{(n)} = 1_{S_n}$. Let us assume that χ_μ is known for $\mu > \lambda$. Since χ_λ occurs only once in φ_λ, $\chi_\lambda = \varphi_\lambda - \sum_{\mu>\lambda} (\varphi_\lambda, \chi_\mu)_{S_n} \chi_\mu$. According to Exercise 29, it is also $(\varphi_\lambda, \chi_{(n)})_{S_n} = 1$. In particular, $\chi_{(n-1,1)} = \varphi_{(n-1,1)} - 1_{S_n}$.

Lemma 11.13. *For every partition λ of n, $\chi_{\lambda'} = \mathrm{sgn} \cdot \chi_\lambda$ holds.*

Proof. According to Exercise 6, $\mathrm{sgn} \cdot \chi_\lambda$ is an irreducible constituent of $\mathrm{sgn} \cdot \varphi_\lambda = \psi_\lambda = \psi_{\lambda''}$ and of $\mathrm{sgn} \cdot \psi_{\lambda'} = \varphi_{\lambda'}$. □

Example 11.14. Let $n = 5$. The partitions of 5 in descending order are (5), $(4,1)$, $(3,2)$, $(3,1^2)$, $(2^2,1)$, $(2,1^3)$, (1^5). Because of Lemma 11.13, we only need to calculate φ_λ for three partitions:

g	(1^5)	$(2,1^3)$	$(2^2,1)$	$(3,1^2)$	$(3,2)$	$(4,1)$	(5)
$\lvert S_5 : \mathrm{C}_{S_5}(g)\rvert$	1	10	15	20	20	30	24
$\varphi_{(4,1)}$	5	3	1	2	.	1	.
$\varphi_{(3,2)}$	10	4	2	1	1	.	.
$\varphi_{(3,1^2)}$	20	6	.	2	.	.	.

The second row contains the lengths of the conjugacy classes and zeros are marked by dots. Now $\chi_{(5)} = 1_{S_5}$. From Remark 11.12 it follows that $\chi_{(4,1)} = \varphi_{(4,1)} - 1_{S_5}$. Furthermore $(\varphi_{(3,2)}, \chi_{(4,1)})_{S_5} = 1$ and $\chi_{(3,2)} = \varphi_{(3,2)} - 1_{S_n} - \chi_{(4,1)}$. Finally $(\varphi_{(3,1^2)}, \chi_{(3,2)})_{S_5} = 1$ and $(\varphi_{(3,1^2)}, \chi_{(4,1)})_{S_5} = 2$. Thus $\chi_{(3,1^2)} = \varphi_{(3,1^2)} - 1_{S_5} - \chi_{(3,2)} - 2\chi_{(4,1)}$.

S_5	(1^5)	$(2,1^3)$	$(2^2,1)$	$(3,1^2)$	$(3,2)$	$(4,1)$	(5)
(5)	1	1	1	1	1	1	1
$(4,1)$	4	2	.	1	-1	.	-1
$(3,2)$	5	1	1	-1	1	-1	.
$(3,1^2)$	6	.	-2	.	.	.	1
$(2^2,1)$	5	-1	1	-1	-1	1	.
$(2,1^3)$	4	-2	.	1	1	.	-1
(1^5)	1	-1	1	1	-1	-1	1

Remark 11.15.

(i) Let λ be a partition of n with Young diagram Y. For a box $b := (i,j)$ of Y, the *hook* $H(b)$ of b is the union of the boxes (i,j), $(i,j+1)$, $\ldots$ and the boxes $(i+1,j)$, $(i+2,j)$, $\ldots$. Let $h(b) := |H(b)| = \lambda_i + \lambda'_j - i - j + 1$. We can write the $h(b)$ into the Young diagram, for example:

7	5	2	1
4	2		
3	1		
1			

The *hook length formula* holds:

$$\chi_\lambda(1) = \frac{n!}{\prod\limits_{b \text{ box of } Y} h(b)}$$

(without proof). For the above tableau, we obtain $\chi_\lambda(1) = 216$.

(ii) If one removes the hook $H(b)$ from Y, one obtains the Young diagram of a partition $\lambda \setminus H(b)$ of $n - h(b)$. Furthermore, $l(b) := \lambda'_j - i$ is called the *leg length* of b. Now let $x \in S_n$ be of type $\mu = (\mu_1, \ldots, \mu_l)$ and $y \in S_{n-\mu_k}$ of type $(\mu_1, \ldots, \mu_{k-1}, \mu_{k+1}, \ldots, \mu_l)$. Then the *Murnaghan-Nakayama formula* holds:

$$\chi_\lambda(x) = \sum_{\substack{b \text{ box of } Y,\\ h(b)=\mu_k}} (-1)^{l(b)} \chi_{\lambda \setminus H(b)}(y)$$

(without proof). In the case $\mu_k = n$, one considers $\chi_{\lambda \setminus H(b)}$ as the trivial character on the trivial group $S_0 = \mathrm{Sym}(\varnothing)$. For $\mu_k = 1$, one obtains the *branching rule*

$$(\chi_\lambda)_{S_{n-1}} = \sum_{\substack{b \text{ box of } Y,\\ h(b)=1}} \chi_{\lambda \setminus H(b)}$$

(without proof).

(iii) In the following, we will construct $\mathrm{Irr}(A_n)$ from $\mathrm{Irr}(S_n)$ and assume that $\mathrm{Irr}(S_n)$ is given. wlog. let always $n \geq 3$.

(iv) Let $\sigma = (a_1, \ldots, a_m) \in S_n$ be a cycle of length m. Then $\sigma = (a_1, a_2)(a_2, a_3) \ldots (a_{m-1}, a_m)$ and $\mathrm{sgn}(\sigma) = (-1)^{m-1}$. If $g \in S_n$ is an arbitrary element of cycle type $(\lambda_1, \ldots, \lambda_k)$, then $\mathrm{sgn}(g) = (-1)^{\lambda_1+\ldots+\lambda_k-k} = (-1)^{n-k}$.

Theorem 11.16. *Let $g \in K \in \mathrm{Cl}(A_n)$ be an element of cycle type $(\lambda_1, \ldots, \lambda_k)$. $K \notin \mathrm{Cl}(S_n)$ if and only if the λ_i are odd and pairwise distinct. If applicable, ${}^{(1,2)}K \in \mathrm{Cl}(A_n)$ and $K \,\dot\cup\, {}^{(1,2)}K \in \mathrm{Cl}(S_n)$.*

Proof. It holds that $K \in \mathrm{Cl}(S_n)$ if and only if $|A_n : \mathrm{C}_{A_n}(g)| = |K| = |S_n : \mathrm{C}_{S_n}(g)|$. Because of $|A_n : \mathrm{C}_{A_n}(g)| = |A_n : A_n \cap \mathrm{C}_{S_n}(g)| = |A_n \mathrm{C}_{S_n}(g) : \mathrm{C}_{S_n}(g)|$ it follows that

$$K \notin \mathrm{Cl}(S_n) \iff \mathrm{C}_{S_n}(g) \subseteq A_n.$$

Let $g = \sigma_1 \ldots \sigma_k$ with disjoint cycles σ_i of length λ_i. Assume first that one λ_i is even. Then $\sigma_i \in \mathrm{C}_{S_n}(g) \setminus A_n$. Thus we can assume that all λ_i are odd. Next, assume that the λ_i are not pairwise distinct, so wlog. $\lambda_1 = \lambda_2$. We write $\sigma_1 = (a_1, \ldots, a_{\lambda_1})$ and $\sigma_2 = (b_1, \ldots, b_{\lambda_1})$. Then $\tau := (a_1, b_1)(a_2, b_2) \ldots (a_{\lambda_1}, b_{\lambda_1}) \notin A_n$. Furthermore, $\tau\sigma_1\tau = \sigma_2$, $\tau\sigma_2\tau = \sigma_1$ and $\tau\sigma_i\tau = \sigma_i$ for all $i \geq 3$. This shows $\tau \in \mathrm{C}_{S_n}(g) \setminus A_n$.

Conversely, let the λ_i be odd and pairwise distinct. We calculate $|S_n : \mathrm{C}_{S_n}(g)|$ by counting the possibilities for g (i. e. the elements of cycle type $(\lambda_1, \ldots, \lambda_k)$). For the λ_1 elements of the cycle σ_1 there are $\frac{n!}{(n-\lambda_1)!}$ possibilities. However, λ_1 possibilities each yield the same element. Thus there are $\frac{n!}{\lambda_1(n-\lambda_1)!}$ possibilities for σ_1. Analogously, there are then $\frac{(n-\lambda_1)!}{\lambda_2(n-\lambda_1-\lambda_2)!}$ possibilities for σ_2 etc. This shows

$$|S_n : \mathrm{C}_{S_n}(g)| = \frac{n!(n-\lambda_1)!(n-\lambda_1-\lambda_2)! \ldots (n-\lambda_1-\ldots-\lambda_{k-1})!}{\lambda_1 \ldots \lambda_k (n-\lambda_1)!(n-\lambda_1-\lambda_2)! \ldots \underbrace{(n-\lambda_1-\ldots-\lambda_k)}_{=0}!} = \frac{n!}{\lambda_1 \ldots \lambda_k}$$

and $|\mathrm{C}_{S_n}(g)| = \lambda_1 \ldots \lambda_k$. Obviously $\langle\sigma_1\rangle \ldots \langle\sigma_k\rangle \subseteq \mathrm{C}_{S_n}(g)$ and $|\langle\sigma_1\rangle \ldots \langle\sigma_k\rangle| = |\langle\sigma_1\rangle| \ldots |\langle\sigma_k\rangle| = \lambda_1 \ldots \lambda_k$. Thus $\mathrm{C}_{S_n}(g) = \langle\sigma_1\rangle \ldots \langle\sigma_k\rangle \subseteq A_n$. This proves the first statement.

Now let $K \notin \mathrm{Cl}(S_n)$. As usual, ${}^{(1,2)}K \in \mathrm{Cl}(A_n)$. Let $K \subseteq \widetilde{K} \in \mathrm{Cl}(S_n)$. Then certainly $K \cup {}^{(1,2)}K \subseteq \widetilde{K}$. Conversely, for every $h \in \widetilde{K}$ there exists an $x \in S_n$ with $h = xgx^{-1}$. If $x \in A_n$, then $h \in K$. Otherwise $(1,2)x \in A_n$ and $h = (1,2)\big((1,2)xgx^{-1}(1,2)\big)(1,2) \in {}^{(1,2)}K$. This shows $\widetilde{K} = K \,\dot\cup\, {}^{(1,2)}K$. □

Example 11.17. For $n \geq 3$ there is always a $K \in \mathrm{Cl}(A_n)$ with $K \notin \mathrm{Cl}(S_n)$. For odd n one can choose cycle type (n) in Theorem 11.16 and for even n cycle type $(n-1, 1)$.

Theorem 11.18. *Let $\chi := \chi_\lambda \in \mathrm{Irr}(S_n)$ for a partition λ of n. In the case $\lambda \neq \lambda'$, $\chi_{A_n} \in \mathrm{Irr}(A_n)$ and in the case $\lambda = \lambda'$, $\chi_{A_n} = \xi_\lambda + {}^{(1,2)}\xi_\lambda$ with $\xi_\lambda \in \mathrm{Irr}(A_n)$ and $\xi_\lambda \neq {}^{(1,2)}\xi_\lambda$. In this way, every irreducible character of A_n occurs.*

Proof. It holds that

$$(\chi, \chi)_{S_n} + (\chi, \mathrm{sgn} \cdot \chi)_{S_n} = \frac{1}{n!} \sum_{g \in S_n} \chi(g)^2 + \mathrm{sgn}(g)\chi(g)^2 = \frac{2}{n!} \sum_{g \in A_n} \chi(g)^2 = (\chi_{A_n}, \chi_{A_n})_{A_n}.$$

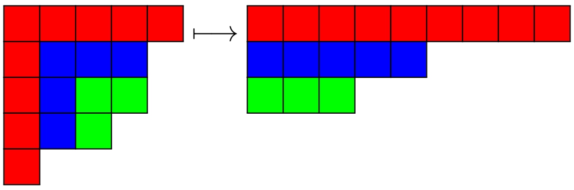
In the case $\lambda \neq \lambda'$, $\mathrm{sgn} \cdot \chi = \chi_{\lambda'} \neq \chi$ and $(\chi_{A_n}, \chi_{A_n})_{A_n} = 1$, i. e. $\chi_{A_n} \in \mathrm{Irr}(A_n)$.

Now let $\lambda = \lambda'$. Then it follows that $(\chi_{A_n}, \chi_{A_n})_{A_n} = 2$ and χ_{A_n} is the sum of two distinct irreducible characters of A_n. From Theorem 4.11 it thus follows that $\chi_{A_n} = \xi_\lambda + {}^{(1,2)}\xi_\lambda$ for some $\xi_\lambda \in \mathrm{Irr}(A_n)$.

Finally, let $\xi \in \mathrm{Irr}(A_n)$ be arbitrary. Then $(\chi_{A_n}, \xi)_{A_n} = (\chi, \xi^{S_n})_{S_n} \neq 0$ for some $\chi \in \mathrm{Irr}(S_n)$. Thus ξ arises in the manner described above. □

Remark 11.19. As usual, $\langle (1,2) \rangle$ acts on $\mathrm{Irr}(A_n)$ and on $\mathrm{Cl}(A_n)$. According to Brauer's permutation lemma, there are exactly as many pairs ${}^{(1,2)}\xi_\lambda \neq \xi_\lambda \in \mathrm{Irr}(A_n)$ as there are pairs ${}^{(1,2)}K \neq K \in \mathrm{Cl}(A_n)$. This can also be seen through the following bijection:

$$\begin{aligned} \Phi : \{\text{partitions } \lambda = \lambda'\} &\longrightarrow \{\lambda = (\lambda_1, \ldots, \lambda_k) \text{ with } \lambda_i \text{ odd and pairwise distinct}\}, \\ (\lambda_1, \ldots, \lambda_k) &\longmapsto (2\lambda_1 - 1, 2\lambda_2 - 3, 2\lambda_3 - 5, \ldots) \end{aligned}$$



Theorem 11.20. *Let $\lambda = \lambda'$ be a partition of n. Then there exists exactly one partition $\mu = (\mu_1, \ldots, \mu_k)$ of n with $\chi_\lambda(g) \equiv 1 \pmod 2$ for some $g \in S_n$ of cycle type μ. It holds that*

$$\xi_\lambda(h) = \begin{cases} \frac{1}{2}\chi_\lambda(h) & h \text{ does not have cycle type } \mu, \\ \frac{1}{2}\chi_\lambda(h) + \frac{1}{2}\sqrt{(-1)^{\frac{n-k}{2}}\mu_1 \ldots \mu_k} & h \text{ is conjugate to } g \text{ in } A_n, \\ \frac{1}{2}\chi_\lambda(h) - \frac{1}{2}\sqrt{(-1)^{\frac{n-k}{2}}\mu_1 \ldots \mu_k} & h \text{ is conjugate to } {}^{(1,2)}g \text{ in } A_n \end{cases}$$

for a suitable choice of ξ_λ. The map $\Gamma\colon \lambda \to \mu$ is a bijection between the partitions $\lambda = \lambda'$ and the partitions μ with pairwise distinct odd parts.

Proof. We argue by induction on (n, λ), where λ is lexicographically ordered. wlog. let $n \geq 4$.

Case 1: $\lambda < (\frac{n-1}{2}, 1, \ldots, 1)$.
Let $\Phi(\lambda) = \eta = (\eta_1, \ldots, \eta_l)$. Then $\eta_1 < n$ and by induction the characters $\xi_1 := \xi_{\Gamma^{-1}(\eta_1)} \in \mathrm{Irr}(A_{\eta_1})$ and $\xi_2 := \xi_{\Gamma^{-1}(\eta_2, \ldots, \eta_l)} \in \mathrm{Irr}(A_{n-\eta_1})$ are known. Thus $\xi_1\xi_2 \in \mathrm{Irr}(A_{\eta_1} \times A_{n-\eta_1})$. By means of $A_{n-\eta_1} \cong \mathrm{Alt}(\{\eta_1 + 1, \eta_1 + 2, \ldots, n\})$ we can consider $A_{\eta_1} \times A_{n-\eta_1}$ as a subgroup of A_n. We set

$$\theta := \left(\xi_1\xi_2 - {}^{(1,2)}\xi_1\xi_2\right)^{A_n}.$$

Let $\theta(h) \neq 0$ for some $h \in A_n$.

Claim: h has cycle type η.
Proof: There exists an $x \in A_n$ with $xhx^{-1} \in A_{\eta_1} \times A_{n-\eta_1}$. We write $xhx^{-1} = (xhx^{-1})_1(xhx^{-1})_2$ with $(xhx^{-1})_1 \in A_{\eta_1}$ and $(xhx^{-1})_2 \in A_{n-\eta_1}$. If $(xhx^{-1})_1$ is not an η_1-cycle, then by the induction hypothesis $\xi_1((xhx^{-1})_1) = {}^{(1,2)}\xi_1((xhx^{-1})_1)$ and $\theta(h) = 0$. We can therefore assume that h_1 is an η_1-cycle. Because $\eta_1 > \eta_i$ for $i \geq 2$, we have $x \in S_{\eta_1} \times S_{n-\eta_1}$. Next, let us assume that h_2 does not have cycle type $(\eta_2, \ldots, \eta_l)$. In the case $\eta_2 = 1$, it would follow that $l = 2$ and $h_2 = 1$, since the η_i are pairwise distinct. This contradiction shows $\eta_2 \neq 1$ and $n - \eta_1 \geq 2$. Now $(xhx^{-1})_2$ also does not have cycle type $(\eta_2, \ldots, \eta_l)$ and it follows that $\xi_2((xhx^{-1})_2) = \xi_2(h)$ for all $x \in S_{\eta_1} \times S_{n-\eta_1}$. Let $\zeta := (\eta_1 + 1, \eta_1 + 2)$. As x varies, $(1,2)\zeta x$ also runs through the set $A_n \cap (S_{\eta_1} \times S_{n-\eta_1})$. Here it holds that

$$\xi_1\big(((1,2)\zeta xhx^{-1}\zeta(1,2))_1\big) - \xi_1((\zeta xhx^{-1}\zeta)_1) = {}^{(1,2)}\xi_1((xhx^{-1})_1) - \xi_1((xhx^{-1})_1).$$

This yields the contradiction

$$\theta(h) = \frac{\xi_2(h_2)}{|A_{\eta_1}||A_{n-\eta_1}|} \sum_{x \in A_n \cap (S_{\eta_1} \times S_{n-\eta_1})} \big(\xi_1((xhx^{-1})_1) - {}^{(1,2)}\xi_1((xhx^{-1})_1)\big) = 0.$$

Claim: $\theta = \widetilde{\xi}_\lambda - {}^{(1,2)}\widetilde{\xi}_\lambda$ with $\widetilde{\xi}_\lambda \in \mathrm{Irr}(A_n)$.
Proof: In the case $\eta_2 = 1$ we have

$$\theta(h) = \frac{1}{|A_{\eta_1}|} \sum_{x \in A_{\eta_1}} \xi_1(xhx^{-1}) - {}^{(1,2)}\xi_1(xhx^{-1}) = \pm\sqrt{(-1)^{\frac{n-l}{2}}\eta_1}$$

according to the induction hypothesis. The sign depends here on the choice of the character ξ_1 and will be irrelevant in the following. Now let $\eta_2 > 1$ and $\chi_2 := \chi_{\Gamma^{-1}(\eta_2,\ldots,\eta_l)} \in \mathrm{Irr}(S_{n-\eta_1})$. The summands in $\theta(h)$ occur in pairs of the form

$$\pm\sqrt{(-1)^{\frac{\eta_1-1}{2}}\eta_1}\left(\frac{1}{2}\chi_2(h_2) \pm \frac{1}{2}\sqrt{(-1)^{\frac{n-\eta_1-(l-1)}{2}}\eta_2\ldots\eta_l}\right)$$

(replace x by $(1,2)\zeta x$ as above). Because $|A_n \cap (S_{\eta_1} \times S_{n-\eta_1})| = 2|A_{\eta_1}||A_{n-\eta_1}|$, it follows that

$$\theta(h) = \pm 2\frac{\sqrt{(-1)^{\frac{\eta_1-1}{2}+\frac{n-\eta_1-l+1}{2}}\eta_1\ldots\eta_l}}{2} = \pm\sqrt{(-1)^{\frac{n-l}{2}}\eta_1\ldots\eta_l}.$$

This shows

$$(\theta,\theta)_{A_n} = \frac{2|A_n : \mathrm{C}_{A_n}(h)|}{|A_n|}\eta_1\ldots\eta_l = 2.$$

Because ${}^{(1,2)}\theta = -\theta$, there exists a $\widetilde{\xi}_\lambda \in \mathrm{Irr}(A_n)$ with $\theta = \widetilde{\xi}_\lambda - {}^{(1,2)}\widetilde{\xi}_\lambda$.

Claim: The statement holds.
Proof: Let $\widetilde{\chi}_\lambda := \widetilde{\xi}_\lambda^{S_n} \in \mathrm{Irr}(S_n)$ (see Theorem 4.13). Let $\widetilde{h} \in A_n$ be of cycle type η. Based on the values calculated above for θ, it follows that

$$\widetilde{\xi}_\lambda(h) = \begin{cases} \frac{1}{2}\widetilde{\chi}_\lambda(h) & h \text{ does not have cycle type } \eta, \\ \frac{1}{2}\widetilde{\chi}_\lambda(h) + \frac{1}{2}\sqrt{(-1)^{\frac{n-l}{2}}\eta_1\ldots\eta_l} & h \text{ is conjugate to } \widetilde{h} \text{ in } A_n, \\ \frac{1}{2}\widetilde{\chi}_\lambda(h) - \frac{1}{2}\sqrt{(-1)^{\frac{n-l}{2}}\eta_1\ldots\eta_l} & h \text{ is conjugate to } {}^{(1,2)}\widetilde{h} \text{ in } A_n \end{cases}$$

for a suitable choice of $\widetilde{\xi}_\lambda$. By definition, $\widetilde{\chi}_\lambda$ vanishes on $S_n \setminus A_n$. If $\widetilde{\chi}_\lambda(h) \equiv 1 \pmod 2$, then $h \in A_n$, and $\frac{1}{2}\widetilde{\chi}_\lambda(h)$ is not an algebraic integer. Thus h must then have cycle type η. Let us assume indirectly that $\widetilde{\chi}_\lambda(\widetilde{h}) \equiv 0 \pmod 2$. Along with $\widetilde{\xi}_\lambda(\widetilde{h})$, then $\frac{1}{2}\sqrt{(-1)^{\frac{n-l}{2}}\eta_1 \ldots \eta_l}$ is also an algebraic integer. Thus

$$\sqrt{(-1)^{\frac{n-l}{2}}\eta_1 \ldots \eta_l} \cdot \frac{1}{2}\sqrt{(-1)^{\frac{n-l}{2}}\eta_1 \ldots \eta_l} = \frac{1}{2}(-1)^{\frac{n-l}{2}}\eta_1 \ldots \eta_l \in \mathbb{Q} \setminus \mathbb{Z}$$

would also be an algebraic integer. This contradiction shows that $\widetilde{\chi}_\lambda$ takes odd values for exactly one conjugacy class of S_n. For different $\lambda < (\frac{n-1}{2}, 1, \ldots, 1)$, one obviously also obtains different $\widetilde{\chi}_\lambda$. Thus the claim of the theorem holds for all characters of S_n except (possibly) one.

Case 2: $\lambda = (\frac{n-1}{2}, 1, \ldots, 1)$.
Because $\Phi(\lambda) = (n)$, n is odd. We denote the "missing" character by $\widetilde{\chi}_\lambda \in \mathrm{Irr}(S_n)$. As usual, $(\widetilde{\chi}_\lambda)_{A_n} = \widetilde{\xi}_\lambda + {}^{(1,2)}\widetilde{\xi}_\lambda$ splits with $\widetilde{\xi}_\lambda \in \mathrm{Irr}(A_n)$. Let $h \in K \in \mathrm{Cl}(A_n)$ be of cycle type $\eta = (\eta_1, \ldots, \eta_l)$. If $K \in \mathrm{Cl}(S_n)$, then $\widetilde{\xi}_\lambda(h) = \widetilde{\xi}_\lambda({}^{(1,2)}h) = {}^{(1,2)}\widetilde{\xi}_\lambda(h) = \frac{1}{2}\widetilde{\chi}_\lambda(h)$. According to Theorem 11.16, we can therefore assume that the η_i are pairwise distinct and odd.

Case 2.1: $\eta \neq (n)$.
Let $\tau := \Phi^{-1}(\eta) \neq \lambda$. According to Case 1, it holds that

$$\begin{aligned}
\widetilde{\xi}_\tau(h)\overline{\widetilde{\xi}_\tau({}^{(1,2)}h)} + \widetilde{\xi}_\tau({}^{(1,2)}h)\overline{\widetilde{\xi}_\tau(h)} &= \frac{1}{2}\widetilde{\chi}_\tau(h)^2 - \frac{1}{2}\eta_1 \ldots \eta_l, \\
\widetilde{\xi}_\tau(h)\overline{\widetilde{\xi}_\tau(h)} + \widetilde{\xi}_\tau({}^{(1,2)}h)\overline{\widetilde{\xi}_\tau({}^{(1,2)}h)} &= \frac{1}{2}\widetilde{\chi}_\tau(h)^2 + \frac{1}{2}\eta_1 \ldots \eta_l.
\end{aligned}$$

Let $a := \widetilde{\xi}_\lambda(h)$ and $b := \widetilde{\xi}_\lambda({}^{(1,2)}h)$. Then $a + b = \widetilde{\chi}_\lambda(h)$. According to the second orthogonality relation,

$$\begin{aligned}
0 &= \sum_{\xi \in \mathrm{Irr}(A_n)} \xi(h)\overline{\xi({}^{(1,2)}h)} = \frac{1}{2}\sum_{\substack{\chi \in \mathrm{Irr}(S_n),\\ \chi_{A_n} \in \mathrm{Irr}(A_n)}} \chi(h)^2 + \Big(\sum_{\substack{\chi \in \mathrm{Irr}(S_n),\\ \chi_{A_n} \notin \mathrm{Irr}(A_n)}} \frac{\chi(h)^2}{2}\Big) - \frac{\eta_1 \ldots \eta_l}{2} + a\overline{b} + b\overline{a} - \frac{\widetilde{\chi}_\lambda(h)^2}{2} \\
&= \frac{1}{2}|\mathrm{C}_{S_n}(h)| - \frac{1}{2}\eta_1 \ldots \eta_l + a\overline{b} + b\overline{a} - \frac{1}{2}\widetilde{\chi}_\lambda(h)^2 = a\overline{b} + b\overline{a} - \frac{1}{2}\widetilde{\chi}_\lambda(h)^2
\end{aligned}$$

and

$$\begin{aligned}
|\mathrm{C}_{A_n}(h)| &= \sum_{\xi \in \mathrm{Irr}(A_n)} \xi(h)\overline{\xi(h)} = \frac{1}{2}\sum_{\substack{\chi \in \mathrm{Irr}(S_n),\\ \chi_{A_n} \in \mathrm{Irr}(A_n)}} \chi(h)^2 + \Big(\sum_{\substack{\chi \in \mathrm{Irr}(S_n),\\ \chi_{A_n} \notin \mathrm{Irr}(A_n)}} \frac{\chi(h)^2}{2}\Big) + \frac{\eta_1 \ldots \eta_l}{2} + a\overline{a} + b\overline{b} - \frac{\widetilde{\chi}_\lambda(h)^2}{2} \\
&= \frac{1}{2}|\mathrm{C}_{S_n}(h)| + \frac{1}{2}\eta_1 \ldots \eta_l + a\overline{a} + b\overline{b} - \frac{1}{2}\widetilde{\chi}_\lambda(h)^2 = |\mathrm{C}_{A_n}(h)| + a\overline{a} + b\overline{b} - \frac{1}{2}\widetilde{\chi}_\lambda(h)^2.
\end{aligned}$$

Equating gives $a\overline{b} + b\overline{a} = a\overline{a} + b\overline{b}$ and $(a - b)(\overline{b} - \overline{a}) = 0$. This shows $a = b = \frac{1}{2}\widetilde{\chi}_\lambda(h)$.

Case 2.2: $\eta = \Phi(\lambda)$.
Then $\widetilde{\xi}_\tau(h) = \frac{1}{2}\widetilde{\chi}_\tau(h)$ for all $\tau \neq \lambda$ according to Case 1. The second orthogonality relation yields

$$\begin{aligned}
0 &= \sum_{\xi \in \mathrm{Irr}(A_n)} \xi(h)\overline{\xi({}^{(1,2)}h)} = \frac{1}{2}\sum_{\substack{\chi \in \mathrm{Irr}(S_n),\\ \chi_{A_n} \in \mathrm{Irr}(A_n)}} \chi(h)^2 + \Big(\sum_{\substack{\chi \in \mathrm{Irr}(S_n),\\ \chi_{A_n} \notin \mathrm{Irr}(A_n)}} \frac{\chi(h)^2}{2}\Big) + a\overline{b} + b\overline{a} - \frac{\widetilde{\chi}_\lambda(h)^2}{2} \\
&= \frac{1}{2}\eta_1 \ldots \eta_l + a\overline{b} + b\overline{a} - \frac{1}{2}\widetilde{\chi}_\lambda(h)^2
\end{aligned}$$

and

$$|\mathrm{C}_{A_n}(h)| = \sum_{\xi\in\mathrm{Irr}(A_n)} \xi(h)\overline{\xi(h)} = \frac{1}{2}\sum_{\substack{\chi\in\mathrm{Irr}(S_n),\\ \chi_{A_n}\in\mathrm{Irr}(A_n)}} \chi(h)^2 + \Big(\sum_{\substack{\chi\in\mathrm{Irr}(S_n),\\ \chi_{A_n}\notin\mathrm{Irr}(A_n)}} \frac{\chi(h)^2}{2}\Big) + a\overline{a} + b\overline{b} - \frac{\widetilde{\chi}_\lambda(h)^2}{2}$$

$$= \frac{1}{2}\eta_1\ldots\eta_l + a\overline{a} + b\overline{b} - \frac{1}{2}\widetilde{\chi}_\lambda(h)^2.$$

Thus $\eta_1\ldots\eta_l = a(\overline{a}-\overline{b}) + b(\overline{b}-\overline{a}) = (a-b)(\overline{a}-\overline{b})$. Let $h := (a_1^1,\ldots,a_{\eta_1}^1)(a_1^2,\ldots,a_{\eta_2}^2)\ldots(a_1^l,\ldots,a_{\eta_l}^l)$ and

$$x = \prod_{i=1}^{l}\prod_{j=1}^{\frac{\eta_i-1}{2}} (a_{j+1}^i, a_{\eta_i-j+1}^i).$$

Then $xhx^{-1} = h^{-1}$. In the case $n-l\equiv 0 \pmod 4$, $\mathrm{sgn}(x) = (-1)^{\frac{\eta_1-1}{2}+\ldots+\frac{\eta_l-1}{2}} = (-1)^{\frac{n-l}{2}} = 1$. Then h and h^{-1} are conjugate in A_n and $a,b\in\mathbb{R}$ (cf. Example 6.15). It follows that $a-b=\pm\sqrt{\eta_1\ldots\eta_l}$ and $a = \frac{1}{2}\widetilde{\chi}_\lambda(h) + \frac{1}{2}\sqrt{\eta_1\ldots\eta_l}$. Finally, let $n-l\equiv 2 \pmod 4$. Here $\mathrm{sgn}(x) = -1$. If there were a $y\in A_n$ with $yhy^{-1} = h^{-1}$, then $x\in x\,\mathrm{C}_{S_n}(h) = y\,\mathrm{C}_{S_n}(h)\subseteq A_n$ (see proof of Theorem 11.16). Thus h and h^{-1} are not conjugate in A_n and $b=\overline{a}$. Thus $a-\overline{a} = \sqrt{-\eta_1\ldots\eta_l}$ and $a = \frac{1}{2}\widetilde{\chi}_\lambda(h) + \frac{1}{2}\sqrt{-\eta_1\ldots\eta_l}$ for a suitable choice of $\widetilde{\xi}_\lambda$. For a fixed $\widetilde{h}\in A_n$ of cycle type η, it therefore holds that

$$\widetilde{\xi}_\lambda(h) = \begin{cases} \frac{1}{2}\widetilde{\chi}_\lambda(h) & h \text{ does not have cycle type } \eta,\\ \frac{1}{2}\widetilde{\chi}_\lambda(h) + \frac{1}{2}\sqrt{(-1)^{\frac{n-l}{2}}\eta_1\ldots\eta_l} & h \text{ is conjugate to } \widetilde{h} \text{ in } A_n,\\ \frac{1}{2}\widetilde{\chi}_\lambda(h) - \frac{1}{2}\sqrt{(-1)^{\frac{n-l}{2}}\eta_1\ldots\eta_l} & h \text{ is conjugate to } {}^{(1,2)}\widetilde{h} \text{ in } A_n.\end{cases}$$

Together with Case 1, it also follows that Γ is a bijection. □

Example 11.21. For $n=5$ one obtains from Example 11.14 the character table of A_5 (cf. Exercise 8):

A_5	(1^5)	$(2^2,1)$	$(3,1^2)$	$(5)_1$	$(5)_2$
(5)	1	1	1	1	1
$(4,1)$	4	.	1	-1	-1
$(3,2)$	5	1	-1	.	.
$(3,1^2)_1$	3	-1	.	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
$(3,1^2)_2$	3	-1	.	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

Remark 11.22. Let $\lambda=\lambda'$ with Young diagram Y and $\Phi(\lambda)=\mu=(\mu_1,\ldots,\mu_k)$. Let $x\in S_n$ be of cycle type μ and let $y\in S_{n-\mu_1}$ be of cycle type $(\mu_2,\ldots,\mu_k)$. We show $\chi_\lambda(x) = (-1)^{\frac{n-k}{2}}$ by induction on k. According to the Murnaghan-Nakayama formula, it holds that

$$\chi_\lambda(x) = \sum_{\substack{b \text{ box of } Y,\\ h(b)=\mu_1}} (-1)^{l(b)}\chi_{\lambda\setminus H(b)}(y) = (-1)^{l(1,1)}\chi_{\lambda\setminus H(1,1)}(y) = (-1)^{\frac{\mu_1-1}{2}}\chi_{\lambda\setminus H(1,1)}(y).$$

In the case $k=1$, we have $\chi_{\lambda\setminus H(1,1)}(y) = 1 = \mu_1$ and $\chi_\lambda(x) = (-1)^{\frac{n-1}{2}}$. Now let $k\geq 2$ and the statement be already proven for $k-1$. Then

$$\chi_\lambda(x) = (-1)^{\frac{\mu_1-1}{2}}\chi_{\lambda\setminus H(1,1)}(y) = (-1)^{\frac{\mu_1-1}{2}+\frac{n-\mu_1-k+1}{2}} = (-1)^{\frac{n-k}{2}}.$$

In particular, $\chi_\lambda(x)$ is odd and in Theorem 11.20 we have $\Gamma=\Phi$. The characters $\widetilde{\chi}_\lambda$ constructed in the proof thus coincide with χ_λ. Furthermore, one thus knows the only non-integer values in the character table of A_n explicitly.

12 Exercises

Exercise 1 (2 points)**.** Determine the irreducible representations of a finite cyclic group.

Exercise 2 (2+2+2 points)**.** Let G be a finite group and V a $\mathbb{C}$-vector space with basis $\{v_g : g \in G\}$. For $g \in G$ let $\Delta(g)$ be the linear map on V that maps v_h to v_{gh} for $h \in G$.

(i) Show that $\Delta\colon G \to \mathrm{GL}(V)$ is a faithful representation of G.

(ii) Calculate the character of Δ.

(iii) Show that Δ is not irreducible for $G \neq 1$.

(One calls Δ the *regular* representation of G.)

Exercise 3 (2+2+2+2+2+2 points)**.** Let $n \geq 3$ and

$$\sigma := \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \qquad \tau := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $\zeta := e^{2\pi i/n}$. One calls $D_{2n} := \langle \sigma, \tau \rangle$ *dihedral group*.

(i) Show: $|D_{2n}| = 2n$.

(ii) Show that an irreducible character χ of D_{2n} with $\chi(\sigma) = 1$ and $\chi(\tau) = -1$ exists.

(iii) Show that for even n, irreducible characters χ and ψ of D_{2n} with $\chi(\sigma) = \psi(\sigma) = -1$ and $\chi(\tau) = -\psi(\tau) = 1$ exist.

(iv) Show that D_{2n} possesses a representation Δ_r with $\Delta_r(\sigma) = \sigma^r$ and $\Delta_r(\tau) = \tau$ for $r \in \mathbb{Z}$.

(v) Show that Δ_r is irreducible for $r = 1, \ldots, (n-1)/2$ (resp. $r = 1, \ldots, (n-2)/2$) if n is odd (resp. even).

(vi) Determine a system of representatives for the similarity classes of irreducible representations of D_{2n}.

Exercise 4 (2 points)**.** Let Δ be a matrix representation of a finite group G, and let $g \in G$. Show that $\Delta(g)$ is diagonalizable.
Hint: One can use the Jordan normal form or Theorem 1.10 of the lecture.

Exercise 5 (2+2+2 points)**.** Let Δ be a matrix representation of G with character χ.

(i) Show that $\overline{\Delta}$ with $\overline{\Delta}(g) := \overline{\Delta(g)}$ for $g \in G$ is also a matrix representation of G. Here $\overline{\Delta(g)}$ is the complex conjugate of $\Delta(g)$.

(ii) Δ is irreducible if and only if $\overline{\Delta}$ is irreducible.

(iii) $\overline{\Delta}$ has character $\overline{\chi}$ with $\overline{\chi}(g) := \overline{\chi(g)} = \chi(g^{-1})$ for $g \in G$.
Hint: One can use Exercise 4.

Exercise 6 (2 points)**.** Let $\chi, \psi \in \mathrm{Irr}(G)$ with $\chi(1) = 1$. Show: $\chi\psi \in \mathrm{Irr}(G)$.

Exercise 7 (2+2+2+2 points)**.** Let G, H be finite, abelian groups. Show:

(i) $\widehat{G} := \mathrm{Irr}(G)$ is an abelian group with respect to multiplication. (One calls $\widehat{G}$ *character group* of G.)

(ii) $\widehat{\widehat{G}} \cong G$
Hint: Think of the bidual space. Do not use (iv).

(iii) $\widehat{G \times H} \cong \widehat{G} \times \widehat{H}$.

(iv) $\widehat{G} \cong G$.

Exercise 8 (2 points)**.** Determine the character table of A_5.

Exercise 9 (2+2+2 points)**.** Show:

(i) For characters χ, ψ of G, the following holds: $\mathrm{Ker}(\chi + \psi) = \mathrm{Ker}(\chi) \cap \mathrm{Ker}(\psi)$.

(ii) Every normal subgroup of G is the kernel of a character.

(iii) $\bigcap_{\chi \in \mathrm{Irr}(G)} \mathrm{Ker}(\chi) = 1$.

Exercise 10 (2 points)**.** Find a monic, integral polynomial with root $\sqrt{2} + \sqrt[3]{3}$.

Exercise 11 (2 points)**.** Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$. Show that $(\chi_N)^G = \chi\rho$ holds, where ρ is the inflation of the regular character of G/N.

Exercise 12 (3 points)**.** Let A be an abelian subgroup of G, and let $\chi \in \mathrm{Irr}(G)$. Show: $\chi(1) \le |G : A|$.

Exercise 13 (3 points)**.** Calculate the eigenvalues of a permutation matrix. (A permutation matrix contains exactly one 1 in each row and each column and zeros otherwise.)

Exercise 14 (3 points)**.** Let $H \le G$ and Δ be a representation of H with character χ. Let $t_1, \ldots, t_m$ be a transversal for the left cosets of H in G. For $g \in G$, let $\dot{\Delta}(g) := \Delta(g)$ if $g \in H$ and $0 \in \mathbb{Z}^{\chi(1)\times\chi(1)}$ otherwise. For $g \in G$, we define the block matrix

$$\Delta^G(g) := \begin{pmatrix} \dot{\Delta}(t_1^{-1} g t_1) & \cdots & \dot{\Delta}(t_1^{-1} g t_m) \\ \vdots & \ddots & \vdots \\ \dot{\Delta}(t_m^{-1} g t_1) & \ldots & \dot{\Delta}(t_m^{-1} g t_m) \end{pmatrix}.$$

Show that Δ^G is a representation of G with character χ^G.

Exercise 15 (3 points)**.** Let K be a finite field with $|K| > 2$. For $a \in K^\times$ and $b \in K$, let $f_{a,b} : K \to K$, $x \mapsto ax + b$. Show that

$$\mathrm{Aff}(1, K) := \{f_{a,b} : a \in K^\times, b \in K\} \le \mathrm{Sym}(K)$$

is a Frobenius group.

Exercise 16 (2 points)**.** Let G be a Frobenius group with Frobenius complement H. Show: $\gcd(|H|, |G : H|) = 1$ (i. e. H is a *Hall subgroup* of G).

Exercise 17 (4 points)**.** Show that subgroups and factor groups of p-elementary (resp. p-quasielementary) groups are again p-elementary (resp. p-quasielementary).

Exercise 18 (2 points)**.** Let P be a finite p-group. Show that P possesses a faithful, irreducible character if and only if $\mathrm{Z}(P)$ is cyclic.

Exercise 19 (3 points)**.** Let $G = \langle g \rangle \cong C_3$ and $\Delta \colon G \to \mathrm{GL}(2, \mathbb{Q})$ with $\Delta(g) = \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Show that Δ is an irreducible $\mathbb{Q}$-representation. Also show that Δ considered as a ($\mathbb{C}$-)representation is reducible.

Exercise 20 (2+2+2 points)**.** Let $G := \mathrm{SL}(2, 3)$. Show:

(i) $|G| = 24$.

(ii) $G/\mathrm{Z}(G) \cong A_4$.
Hint: Consider the action of G on the set of one-dimensional subspaces of $\mathbb{F}_3^2$.

(iii) G is not an M-group. (Thus, not every solvable group is an M-group.)

Exercise 21 (3 points)**.** Let G be a group of odd order. Show: $k(G) \equiv |G| \pmod{16}$.
Hint: Consider $\overline{\chi} = \chi \in \mathrm{Irr}(G)$.

Exercise 22 (3 points)**.** Construct irreducible characters with Frobenius-Schur indicator 0, 1 and -1.

Exercise 23 (10 points)**.** Decide which of the following group properties can be determined from the character table of G:

(i) $|G|$,

(ii) commutativity,

(iii) the lengths of the conjugacy classes,

(iv) simplicity,

(v) normal subgroups and their order,

(vi) isomorphism type of G/G',

(vii) isomorphism type of $\mathrm{Z}(G)$,

(viii) solvability,

(ix) isomorphism type (of G),

(x) derived length, i. e. $\min\{n \in \mathbb{N} : G^{(n)} = 1\}$.
Hint: Use Google.[3]

[3] For writing final theses, researching must also be practiced :-)

Bonus: nilpotency, nilpotency class, $\mathrm{F}(G)$ (Fitting group), $\Phi(G)$ (Frattini group), ... (as far as terms are known).

Exercise 24 (5 points)**.** Given is the character table of G:

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	−1	1	1	1	−1	1	1	1	−1	1	1	1	−1	1	1
1	−1	1	1	$\overline{\beta}$	$-\overline{\beta}$	$\overline{\beta}$	$\overline{\beta}$	β	$-\beta$	β	β	1	−1	1	1
1	−1	1	1	β	$-\beta$	β	β	$\overline{\beta}$	$-\overline{\beta}$	$\overline{\beta}$	$\overline{\beta}$	1	−1	1	1
1	1	1	1	$\overline{\beta}$	$\overline{\beta}$	$\overline{\beta}$	$\overline{\beta}$	β	β	β	β	1	1	1	1
1	1	1	1	β	β	β	β	$\overline{\beta}$	$\overline{\beta}$	$\overline{\beta}$	$\overline{\beta}$	1	1	1	1
2	.	α	α'	2	.	α	α'	2	.	α	α'	2	.	α	α'
2	.	α'	α	2	.	α'	α	2	.	α'	α	2	.	α'	α
2	.	α	α'	$2\overline{\beta}$	.	γ'	γ	2β	.	$\overline{\gamma'}$	$\overline{\gamma}$	2	.	α	α'
2	.	α	α'	2β	.	$\overline{\gamma'}$	$\overline{\gamma}$	$2\overline{\beta}$	.	γ'	γ	2	.	α	α'
2	.	α'	α	$2\overline{\beta}$	.	γ	γ'	2β	.	$\overline{\gamma}$	$\overline{\gamma'}$	2	.	α'	α
2	.	α'	α	2β	.	$\overline{\gamma}$	$\overline{\gamma'}$	$2\overline{\beta}$	.	γ	γ'	2	.	α'	α
3	−3	3	3	.	.	.	.	.	.	.	.	−1	1	−1	−1
3	3	3	3	.	.	.	.	.	.	.	.	−1	−1	−1	−1
6	.	3α	$3\alpha'$	.	.	.	.	.	.	.	.	−2	.	$-\alpha$	$-\alpha'$
6	.	$3\alpha'$	3α	.	.	.	.	.	.	.	.	−2	.	$-\alpha'$	$-\alpha$

with

$$. = 0, \qquad \alpha = \frac{-1+\sqrt{5}}{2}, \qquad \alpha' = \frac{-1-\sqrt{5}}{2},$$

$$\beta = \frac{-1+\sqrt{3}i}{2}, \qquad \gamma = e^{2\pi i/15} + e^{8\pi i/15}, \qquad \gamma' = e^{14\pi i/15} + e^{-4\pi i/15}.$$

Determine as many properties of G as possible.

Exercise 25 (2 points)**.** Show that a simple group has no irreducible character of degree 2.

Exercise 26 (2 points)**.** Let G be a simple group with an involution x, such that $\mathrm{C}_G(x)$ is cyclic. Determine G.

Exercise 27 (2+2 points)**.** Let $n \in \mathbb{N}$. Show:

(i) Two elements in S_n are conjugate if and only if they have the same cycle type.

(ii) The character table of S_n is integer-valued.
Hint: Use Galois theory.

Exercise 28 (2 points)**.** Let $G = G' \leq \mathrm{GL}(2, \mathbb{C})$ and $A \neq 1$ be an abelian normal subgroup of G. Show $A = \mathrm{Z}(G) \cong C_2$.

Exercise 29 (2+2+2+2+2+2+2 points)**.** Let $f : G \to \mathrm{Sym}(\Omega)$ be a group action on a finite, non-empty set Ω. Show:

(i) It holds that $\mathrm{Sym}(\Omega) \cong S_{|\Omega|}$. In the following, we can therefore assume $\Omega = \{1, \dots, n\}$.

(ii) The map $F : \mathrm{Sym}(\Omega) \to \mathrm{GL}(n, \mathbb{C})$, $\pi \mapsto (\delta_{i\pi(j)})_{i,j=1}^{n}$ is a monomorphism. In particular, $\Delta := F \circ f : G \to \mathrm{GL}(n, \mathbb{C})$ is a representation of G. (One calls Δ a *permutation representation.*)

(iii) For the character χ of Δ, it holds that $\chi(g) := |\{\omega \in \Omega : {}^{g}\omega = \omega\}|$ for $g \in G$. (One calls χ a *permutation character.*)

(iv) Let $\omega_1, \dots, \omega_m$ be a system of representatives for the orbits of f. Then

$$\chi = \sum_{i=1}^{m} 1_{G_{\omega_i}}^{G},$$

where $G_\omega := \{g \in G : {}^{g}\omega = \omega\}$ is the *stabilizer* of $\omega \in \Omega$ in G.

(v) It holds that $m = (1_G, \chi)_G$. In particular, $\chi - 1_G$ is a character of G if $|\Omega| > 1$.

(vi) If $H_1, \dots, H_k$ are arbitrary subgroups of G, then

$$\sum_{i=1}^{k} 1_{H_i}^{G}$$

is also a permutation character of G.

(vii) The set of permutation characters of G is closed under addition and multiplication.

Exercise 30 (2 points)**.** Let $n \in \mathbb{N}$. Show that $\mathrm{GL}(n, \mathbb{C})$ has a subgroup that is isomorphic to S_{n+1}. *Hint:* One can use Exercise 29.

Appendix: Character Tables

The character tables of S_3, S_4, S_5 and A_3, A_4, A_5 have already been calculated.

S_6	(1^6)	$(2,1^4)$	$(2^2,1^2)$	(2^3)	$(3,1^3)$	$(3,2,1)$	(3^2)	$(4,1^2)$	$(4,2)$	$(5,1)$	(6)
(6)	1	1	1	1	1	1	1	1	1	1	1
$(5,1)$	5	3	1	−1	2	.	−1	1	−1	.	−1
$(4,2)$	9	3	1	3	.	.	.	−1	1	−1	.
$(4,1^2)$	10	2	−2	−2	1	−1	1	.	.	.	1
(3^2)	5	1	1	−3	−1	1	2	−1	−1	.	.
$(3,2,1)$	16	.	.	.	−2	.	−2	.	.	1	.
$(3,1^3)$	10	−2	−2	2	1	1	1	.	.	.	−1
(2^3)	5	−1	1	3	−1	−1	2	1	−1	.	.
$(2^2,1^2)$	9	−3	1	−3	.	.	.	1	1	−1	.
$(2,1^4)$	5	−3	1	1	2	.	−1	−1	−1	.	1
(1^6)	1	−1	1	−1	1	−1	1	−1	1	1	−1

A_6	(1^6)	$(2^2,1^2)$	$(3,1^3)$	(3^2)	$(4,2)$	$(5,1)_1$	$(5,1)_2$
(6)	1	1	1	1	1	1	1
$(5,1)$	5	1	2	−1	−1	.	.
$(4,2)$	9	1	.	.	1	−1	−1
$(4,1^2)$	10	−2	1	1	.	.	.
(3^2)	5	1	−1	2	−1	.	.
$(3,2,1)_2$	8	.	−1	−1	.	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
$(3,2,1)_1$	8	.	−1	−1	.	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$

S_7	(1^7)	$(2,1^5)$	$(2^2,1^3)$	$(2^3,1)$	$(3,1^4)$	$(3,2,1^2)$	$(3,2^2)$	$(3^2,1)$	$(4,1^3)$	$(4,2,1)$	$(4,3)$	$(5,1^2)$	$(5,2)$	$(6,1)$	(7)
(7)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$(6,1)$	6	4	2	.	3	1	-1	.	2	.	-1	1	-1	.	-1
$(5,2)$	14	6	2	2	2	.	2	-1	.	.	.	-1	1	-1	.
$(5,1^2)$	15	5	-1	-3	3	-1	-1	.	1	-1	1	.	.	.	1
$(4,3)$	14	4	2	.	-1	1	-1	2	-2	.	1	-1	-1	.	.
$(4,2,1)$	35	5	-1	1	-1	-1	-1	-1	-1	1	-1	.	.	1	.
$(4,1^3)$	20	.	-4	.	2	.	2	2	.	.	.	.	.	.	-1
$(3^2,1)$	21	1	1	-3	-3	1	1	.	-1	-1	-1	1	1	.	.
$(3,2^2)$	21	-1	1	3	-3	-1	1	.	1	-1	1	1	-1	.	.
$(3,2,1^2)$	35	-5	-1	-1	-1	1	-1	-1	1	1	1	.	.	-1	.
$(3,1^4)$	15	-5	-1	3	3	1	-1	.	-1	-1	-1	.	.	.	1
$(2^3,1)$	14	-4	2	.	-1	-1	-1	2	2	.	-1	-1	1	.	.
$(2^2,1^3)$	14	-6	2	-2	2	.	2	-1	.	.	.	-1	-1	1	.
$(2,1^5)$	6	-4	2	.	3	-1	-1	.	-2	.	1	1	1	.	-1
(1^7)	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	-1	1

A_7	(1^7)	$(2^2,1^3)$	$(3,1^4)$	$(3,2^2)$	$(3^2,1)$	$(4,2,1)$	$(5,1^2)$	$(7)_1$	$(7)_2$
(7)	1	1	1	1	1	1	1	1	1
$(6,1)$	6	2	3	-1	.	.	1	-1	-1
$(5,2)$	14	2	2	2	-1	.	-1	.	.
$(5,1^2)$	15	-1	3	-1	.	-1	.	1	1
$(4,3)$	14	2	-1	-1	2	.	-1	.	.
$(4,2,1)$	35	-1	-1	-1	-1	1	.	.	.
$(4,1^3)_1$	10	-2	1	1	1	.	.	$\frac{-1+\sqrt{-7}}{2}$	$\frac{-1-\sqrt{-7}}{2}$
$(4,1^3)_2$	10	-2	1	1	1	.	.	$\frac{-1-\sqrt{-7}}{2}$	$\frac{-1+\sqrt{-7}}{2}$
$(3^2,1)$	21	1	-3	1	.	-1	1	.	.

A_8	(1^8)	$(2^2,1^4)$	(2^4)	$(3,1^5)$	$(3,2^2,1)$	$(3^2,1^2)$	$(4,2,1^2)$	(4^2)	$(5,1^3)$	$(6,2)$	$(5,3)_1$	$(5,3)_2$	$(7,1)_1$	$(7,1)_2$
(8)	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$(7,1)$	7	3	-1	4	.	1	1	-1	2	-1	-1	-1	.	.
$(6,2)$	20	4	4	5	1	-1	.	.	.	.	.	1	-1	-1
$(6,1^2)$	21	1	-3	6	-2	.	-1	1	1	1	1	.	.	.
$(5,3)$	28	4	-4	1	1	1	.	.	-2	1	1	-1	.	.
$(5,2,1)$	64	.	.	4	.	-2	.	.	-1	-1	-1	.	1	1
$(5,1^3)$	35	-5	3	5	1	2	-1	-1	.	.	.	.	.	.
(4^2)	14	2	6	-1	-1	2	.	2	-1	-1	-1	.	.	.
$(4,3,1)$	70	2	-2	-5	-1	1	.	-2	.	.	.	1	.	.
$(4,2^2)$	56	.	8	-4	.	-1	.	.	1	1	1	-1	.	.
$(4,2,1^2)_1$	45	-3	-3	.	.	.	1	1	.	.	.	.	$\frac{-1+\sqrt{-7}}{2}$	$\frac{-1-\sqrt{-7}}{2}$
$(4,2,1^2)_2$	45	-3	-3	.	.	.	1	1	.	.	.	.	$\frac{-1-\sqrt{-7}}{2}$	$\frac{-1+\sqrt{-7}}{2}$
$(3^2,2)_1$	21	1	-3	-3	1	.	-1	1	1	$\frac{-1+\sqrt{-15}}{2}$	$\frac{-1-\sqrt{-15}}{2}$	.	.	.
$(3^2,2)_2$	21	1	-3	-3	1	.	-1	1	1	$\frac{-1-\sqrt{-15}}{2}$	$\frac{-1+\sqrt{-15}}{2}$	.	.	.

Let q be a prime power, $x,y\in\mathbb{F}_q^\times$ and $z\in\mathbb{F}_{q^2}\setminus\mathbb{F}_q$. Every element in $\mathrm{GL}(2,q)$ is conjugate to one of the following matrices: $a_x:=\mathrm{diag}(x,x)$, $b_x:=\left(\begin{smallmatrix}x&1\\0&x\end{smallmatrix}\right)$, $c_{x,y}:=\mathrm{diag}(x,y)$ with $x\ne y$ and d_z with eigenvalues z,z^q. Here $c_{x,y}\sim c_{y,x}$ and $d_z\sim d_{z^q}$ hold. In particular,

$$2q-2+\frac{(q-1)(q-2)}{2}+\frac{q(q-1)}{2}=q^2-1$$

is the number of conjugacy classes of $\mathrm{GL}(2,q)$. Let $\alpha,\beta\in\mathrm{Irr}(\mathbb{F}_q)$ and $\gamma\in\mathrm{Irr}(\mathbb{F}_{q^2})$ with $\alpha\ne\beta$ and $\gamma^q\ne\gamma$. The character table of $\mathrm{GL}(2,q)$ is

#		$q-1$	$q-1$	$(q-1)(q-2)/2$	$q(q-1)/2$
$\lvert {}^G g\rvert$		1	q^2-1	q^2+q	q^2-q
$\mathrm{GL}(2,q)$	#	a_x	b_x	$c_{x,y}$	d_z
χ_α	$q-1$	$\alpha(x)^2$	$\alpha(x)^2$	$\alpha(x)\alpha(y)$	$\alpha(z^{q+1})$
ψ_α	$q-1$	$q\alpha(x)^2$	0	$\alpha(x)\alpha(y)$	$-\alpha(z^{q+1})$
$\rho_{\alpha,\beta}$	$(q-1)(q-2)/2$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y)+\alpha(y)\beta(x)$	0
τ_γ	$q(q-1)/2$	$(q-1)\gamma(x)$	$-\gamma(x)$	0	$-\gamma(z)-\gamma(z)^q$

It holds that $|\mathrm{SL}(2,q)| = (q-1)q(q+1)$. There exist cyclic subgroups $X, Y \leq G$ of order $q-1$ and $q+1$ respectively with $X \cap Y = Z = \langle -1_2 \rangle = \mathrm{Z}(G)$. Let $X_0 := X \setminus Z$ and $Y_0 := Y \setminus Z$. The p-Sylow subgroups are elementary abelian of order q.

First let $q = p^n$ be odd. Up to conjugation there are two elements a, b of order p. Let $\alpha, \alpha_0 \in \mathrm{Irr}(X) \setminus \{1_X\}$ and $\beta, \beta_0 \in \mathrm{Irr}(Y) \setminus \{1_Y\}$ with $\alpha^2 \neq \alpha_0^2 = 1 = \beta_0^2 \neq \beta^2$. The character tables are:

- $q \equiv 1 \pmod 4$:

g	1	-1	$x \in X_0$	$y \in Y_0$	$\pm a$	$\pm b$
$\lvert \mathrm{C}_G(g)\rvert$	1	1	q^2+q	q^2-q	$\frac{q^2-1}{2}$	$\frac{q^2-1}{2}$
1_G	1	1	1	1	1	1
ρ	q	q	1	-1	0	0
η_1	$\frac{q+1}{2}$	$\frac{q+1}{2}$	$\alpha_0(x)$	0	$\frac{1+\sqrt{q}}{2}$	$\frac{1-\sqrt{q}}{2}$
η_1^*	$\frac{q+1}{2}$	$\frac{q+1}{2}$	$\alpha_0(x)$	0	$\frac{1-\sqrt{q}}{2}$	$\frac{1+\sqrt{q}}{2}$
η_2	$\frac{q-1}{2}$	$-\frac{q-1}{2}$	0	$-\beta_0(y)$	$\frac{\mp 1+\sqrt{q}}{2}$	$\frac{\mp 1-\sqrt{q}}{2}$
η_2^*	$\frac{q-1}{2}$	$-\frac{q-1}{2}$	0	$-\beta_0(y)$	$\frac{\mp 1-\sqrt{q}}{2}$	$\frac{\mp 1+\sqrt{q}}{2}$
χ_α	$q+1$	$\alpha(-1)(q+1)$	$\alpha(x)+\alpha(x)^{-1}$	0	$\alpha(\pm 1)$	$\alpha(\pm 1)$
ψ_β	$q-1$	$\beta(-1)(q-1)$	0	$-\beta(y)-\beta(y)^{-1}$	$-\beta(\pm 1)$	$-\beta(\pm 1)$

- $q \equiv -1 \pmod 4$:

g	1	-1	$x \in X_0$	$y \in Y_0$	$\pm a$	$\pm b$
$\lvert \mathrm{C}_G(g)\rvert$	1	1	q^2+q	q^2-q	$\frac{q^2-1}{2}$	$\frac{q^2-1}{2}$
1_G	1	1	1	1	1	1
ρ	q	q	1	-1	0	0
η_1	$\frac{q+1}{2}$	$-\frac{q+1}{2}$	$\alpha_0(x)$	0	$\frac{\mp 1+\sqrt{-q}}{2}$	$\frac{\mp 1-\sqrt{-q}}{2}$
$\overline{\eta_1}$	$\frac{q+1}{2}$	$-\frac{q+1}{2}$	$\alpha_0(x)$	0	$\frac{\mp 1-\sqrt{-q}}{2}$	$\frac{\mp 1+\sqrt{-q}}{2}$
η_2	$\frac{q-1}{2}$	$\frac{q-1}{2}$	0	$-\beta_0(y)$	$\frac{-1+\sqrt{-q}}{2}$	$\frac{-1-\sqrt{-q}}{2}$
$\overline{\eta_2}$	$\frac{q-1}{2}$	$\frac{q-1}{2}$	0	$-\beta_0(y)$	$\frac{-1-\sqrt{-q}}{2}$	$\frac{-1+\sqrt{-q}}{2}$
χ_α	$q+1$	$\alpha(-1)(q+1)$	$\alpha(x)+\alpha(x)^{-1}$	0	$\alpha(\pm 1)$	$\alpha(\pm 1)$
ψ_β	$q-1$	$\beta(-1)(q-1)$	0	$-\beta(y)-\beta(y)^{-1}$	$-\beta(\pm 1)$	$-\beta(\pm 1)$

- $p = 2$: Here $Z = 1$ and up to conjugation there is only one element a of order 2.

g	1	a	$x \in X_0$	$y \in Y_0$
$\lvert \mathrm{C}_G(g)\rvert$	1	q	$q-1$	$q+1$
1_G	1	1	1	1
ρ	q	0	1	-1
χ_α	$q+1$	1	$\alpha(x)+\alpha(x)^{-1}$	0
ψ_β	$q-1$	-1	0	$-\beta(y)-\beta(y)^{-1}$

Here $\chi_\alpha = \chi_{\overline{\alpha}}$ and $\psi_\beta = \psi_{\overline{\beta}}$ hold.

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